10. Linear programs

- linear programs
- piecewise-linear minimization
- geometry of LPs
- standard form LP

Definition

a *linear program* (LP) is an optimization problem where the objective function is linear and the constraint functions are linear or affine

LP general form

$$
\begin{array}{ll}
\text{minimize (or maximize)} & \displaystyle\sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \ldots, m \\
 & \displaystyle\sum_{j=1}^{n} g_{ij} x_j \le h_i, \quad i = 1, \ldots, p
\end{array} \tag{10.1}
$$

- n optimization variables $x_1, \ldots, x_n$
- coefficients $c_j, a_{ij}, g_{ij}, h_i, b_i$ are given

LP in compact form

$$\begin{aligned} \text{minimize (or maximize)} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} \le \boldsymbol{b} \\ & G\boldsymbol{x} \le \boldsymbol{h} \end{aligned} \tag{10.2}$$

- A is an $m \times n$ matrix with entries a_{ij}
- G is an $p \times n$ matrix with entries g_{ij}
- $\boldsymbol{b} = (b_1, \dots, b_m)$
- $\boldsymbol{h} = (h_1, \dots, h_p)$
- $\boldsymbol{c} = (c_1, \dots, c_n)$

Example: the diet problem

- we need to create a meal that contains at least 12 units of protein, 9 units of iron, and 15 units of thiamine;
- we have two main foods A and B:
  - each gram of A contains 2 units of protein, 1 unit of iron, and 1 unit of thiamine; each gram B contains 1 unit of protein, 1 unit of iron, and 3 units of thiamine
  - each gram of A costs 30 cents, while each gram of B costs 40 cents
- how many grams of each of the food should be used to minimize the cost of the meal?

let x_1 and x_2 be the number of grams of food A and B used in the meal; then, the problem can formulated as

$$\begin{array}{ll} \text{minimize} & 30x_1 + 40x_2 \\ \text{subject to} & 2x_1 + x_2 \geq 12 \\ & x_1 + x_2 \geq 9 \\ & x_1 + 3x_2 \geq 15 \\ & x_1, x_2 \geq 0 \end{array}$$

Example: alloy mixture

we want to create a new alloy consisting of 40% iron, 35% nickel, and 25% cobalt from a mixture of several available alloys that have the metal properties listed in the below table

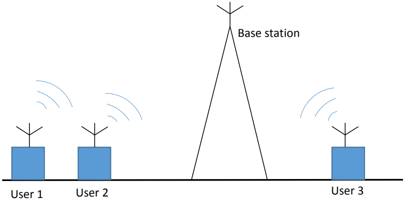
property	alloy 1	alloy 2	alloy 3	alloy 4
% of iron	70	25	40	20
% of nickel	10	15	50	50
% of cobalt	20	60	10	30
cost ($/kg)	22	18	25	24

we want to determine the proportions of these alloys that should be blended together so that we produce the new alloy at a minimum cost

- let x_i, $i = 1, 2, 3, 4$, be the proportion of alloy i that is used to produce the new alloy
- the problem can be formulated as

$$
\begin{array}{ll}
\text{minimize} & 22x_1 + 18x_2 + 25x_3 + 24x_4 \\
\text{subject to} & 0.7x_1 + 0.25x_2 + 0.4x_3 + 0.2x_4 = 0.4 \\
& 0.1x_1 + 0.15x_2 + 0.5x_3 + 0.5x_4 = 0.35 \\
& 0.2x_1 + 0.6x_2 + 0.1x_3 + 0.3x_4 = 0.25 \\
& x_1 + x_2 + x_3 + x_4 = 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Example: wireless communication



- n "mobile" users; user i transmits a signal to the base station with power p_i and an attenuation factor of β_i (*i.e.*, signal power received at the base station from user i is $\beta_i p_i$)
- total power received from all other users is considered interference (*i.e.*, the interference for user i is $\sum_{j \neq i} \beta_j p_j$)
- for the communication with user i to be reliable, the signal-to-interference ratio must exceed a threshold γ_i
- we are interested in minimizing the total power transmitted by all users subject to having reliable communications for all users

Problem formulation

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} p_i \\ \text{subject to} & \frac{\beta_i p_i}{\sum_{j\neq i} \beta_j p_j} \geq \gamma_i, \quad i = 1, \ldots, n \\ & p_i \geq 0, \quad i = 1, \ldots, n \end{array}$$

LP formulation

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} p_i \\ \text{subject to} & \beta_i p_i - \gamma_i \sum_{j\neq i} \beta_j p_j \geq 0, \quad i = 1, \ldots, n \\ & p_i \geq 0, \quad i = 1, \ldots, n \end{array}$$

Example: manufacturing problem

we have n products and m raw materials, and

- c_j is profit of product j;
- b_i are the available units of material i;
- a_{ij} is the number of units of material i product j needs in order to be produced

the manufacturing problem is to choose the amount of product j produced such that the profit is maximized; we can formulate this problem as:

$$
\begin{array}{ll}
\text{maximize} & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{array}
\tag{10.3}
$$

where the variable x_j represents the amount of product j to be produced

Example: assignment problem

- we want to match N people to N tasks
- each person is assigned to one task (each task assigned to one person)
- cost of assigning person i to task j is c_{ij}
- variable $x_{ij} = 1$ if person i is assigned to task j; $x_{ij} = 0$ otherwise

Combinatorial formulation

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} x_{ij} \\ \text{subject to} & \sum_{i=1}^{N} x_{ij} = 1, \quad j = 1, \ldots, N \\ & \sum_{j=1}^{N} x_{ij} = 1, \quad i = 1, \ldots, N \\ & x_{ij} \in \{0, 1\}, \quad i, j = 1, \ldots, N \end{array}$$

$N!$ possible assignments (*e.g.*, $10! = 3628800$)

LP formulation

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}x_{ij} \\ \text{subject to} & \sum_{i=1}^{N} x_{ij} = 1, \quad j = 1,\ldots,N \\ & \sum_{j=1}^{N} x_{ij} = 1, \quad i = 1,\ldots,N \\ & 0 \le x_{ij} \le 1, \quad i,j = 1,\ldots,N \end{array}$$

- we have *relaxed* the constraints $x_{ij} \in \{0,1\}$
- it can be shown that the solution $x_{ij}^{\star} \in \{0,1\}$
- hence, we can solve this hard combinatorial problem efficiently by solving the above LP

Outline

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- **piecewise-linear minimization**
- geometry of LPs
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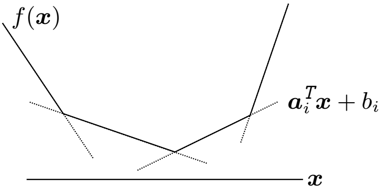
Piecewise-linear minimization

Piecewise-linear function: a function $f : \mathbb{R}^n \to \mathbb{R}$ is *piecewise-linear* if it can be expressed as

$$f(\boldsymbol{x}) = \max_{i=1,\ldots,m} (\boldsymbol{a}_i^T \boldsymbol{x} + b_i)$$



Piecewise-linear minimization

$$\text{minimize} \quad f(\boldsymbol{x}) = \max_{i=1,\ldots,m} (\boldsymbol{a}_i^T \boldsymbol{x} + b_i)$$

Equivalent LP formulation

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \boldsymbol{a}_i^T\boldsymbol{x} + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

(for fixed $\boldsymbol{x}$, the optimal t is $t = f(\boldsymbol{x})$)

Matrix form

$$\begin{array}{rl} \text{minimize} & \tilde{\boldsymbol{c}}^T\tilde{\boldsymbol{x}} \\ \text{subject to} & \tilde{A}\tilde{\boldsymbol{x}} \leq \tilde{\boldsymbol{b}} \end{array}$$

where

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x} \\ t \end{bmatrix}, \quad \tilde{\boldsymbol{c}} = \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \boldsymbol{a}_1^T & -1 \\ \vdots & \vdots \\ \boldsymbol{a}_m^T & -1 \end{bmatrix}, \quad \tilde{\boldsymbol{b}} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \end{bmatrix}$$

ℓ_1-Norm approximation

$$\text{minimize} \quad \|A\boldsymbol{x} - \boldsymbol{b}\|_1$$

- $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$
- for a vector $\boldsymbol{y} \in \mathbb{R}^m$, we have

$$\|\boldsymbol{y}\|_1 = \sum_{i=1}^{m} |y_i| = \sum_{i=1}^{m} \max\{y_i, -y_i\}$$

Equivalent LP formulation

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} u_i \\ \text{subject to} & -\boldsymbol{u} \leq A\boldsymbol{x} - \boldsymbol{b} \leq \boldsymbol{u} \end{array}$$

with variables $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^m$

Robust curve fitting

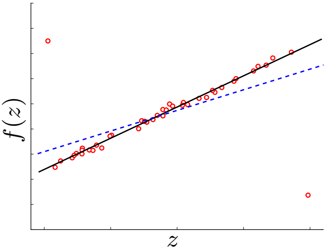
problem of fitting the data points (z_i, y_i) to the straight line $x_1 + x_2 z \approx y$ can be formulated as

$$\text{minimize} \quad \|A\boldsymbol{x} - \boldsymbol{b}\|_1$$

where

$$A = \begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_m \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- red circles represent the data
- blue dotted line from minimizing $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$
- black line from minimizing $\|A\boldsymbol{x} - \boldsymbol{b}\|_1$
- ℓ_1-norm more robust to outliers



Interview scheduling

- a company needs to schedule job interviews for n candidates numbered $1, 2, \ldots, n$
- candidate i is scheduled to be the ith interview
- the starting time of candidate i must be in the interval $[\alpha_i, \beta_i]$, where $\alpha_i < \beta_i$
- the goal is to to find n starting times of interviews so that the minimal starting time difference between consecutive interviews is maximal

let t_i denote the starting time of interview i; the objective function is the minimal difference between consecutive starting times of interviews:

$$f(\boldsymbol{t}) = \min\left\{t_2 - t_1, t_3 - t_2, \ldots, t_n - t_{n-1}\right\},$$

where $\boldsymbol{t} = (t_1, \ldots, t_n)$

Problem formulation

$$\begin{array}{ll} \text{maximize} & \min\left\{t_2 - t_1, t_3 - t_2, \ldots, t_n - t_{n-1}\right\} \\ \text{subject to} & \alpha_i \leq t_i \leq \beta_i, \quad i = 1, 2, \ldots, n, \end{array}$$

with variable $\boldsymbol{t} \in \mathbb{R}^n$

Equivalent LP

$$\begin{array}{ll} \text{maximize} & s \\ \text{subject to} & t_{i+1} - t_i \geq s, \quad i = 1, 2, \ldots, n-1 \\ & \alpha_i \leq t_i \leq \beta_i, \quad i = 1, 2, \ldots, n, \end{array}$$

with variables $\boldsymbol{t} \in \mathbb{R}^n$ and $s \in \mathbb{R}$

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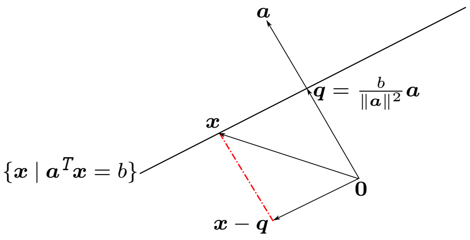
Hyperplane

a *hyperplane* is the solution set $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}^T\boldsymbol{x} = b\}$ with $\boldsymbol{a} \neq \boldsymbol{0}$



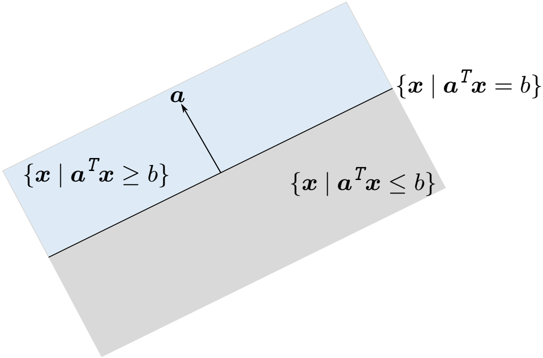
- $\boldsymbol{a}$ is called *normal vector*
- $\boldsymbol{x}$ is in the hyperplane if and only if $\boldsymbol{x} - \boldsymbol{q}$ is orthogonal to $\boldsymbol{a}$:

$$\boldsymbol{a}^T\boldsymbol{x} = b = \boldsymbol{a}^T\boldsymbol{q} \Longrightarrow \boldsymbol{a}^T(\boldsymbol{x} - \boldsymbol{q}) = 0$$

Halfspaces

the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}^T\boldsymbol{x} = b\}$ divides $\mathbb{R}^n$ in two *halfspaces*

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}^T\boldsymbol{x} \leq b\} \quad \text{and} \quad \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}^T\boldsymbol{x} \geq b\}$$



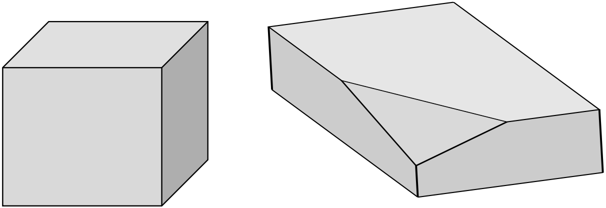
Polyhedron

a *ployhedron* is the intersection of finitely many halfspaces; it is any set of points that satisfies a finite number of inequalities

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m
\end{aligned}
$$

in matrix notation, a polyhedron can be defined as

$$
\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq \boldsymbol{b}\} \tag{10.4}
$$

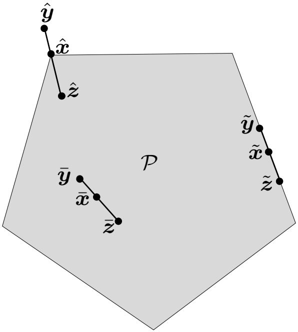


Extreme points

a point $\boldsymbol{x} \in \mathcal{P}$ is an *extreme point* of $\mathcal{P}$ if it *cannot* be written as the convex combination

$$\boldsymbol{x} = \theta \boldsymbol{y} + (1-\theta)\boldsymbol{z}, \quad \theta \in (0,1)$$

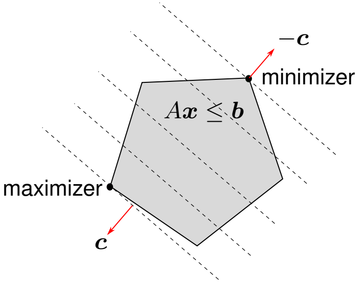
for some $\boldsymbol{y}, \boldsymbol{z} \in \mathcal{P}$



- $\hat{\boldsymbol{x}}$ is an extreme point
- $\bar{\boldsymbol{x}}$ and $\tilde{\boldsymbol{x}}$ are not extreme points

Geometrical interpretation of LP

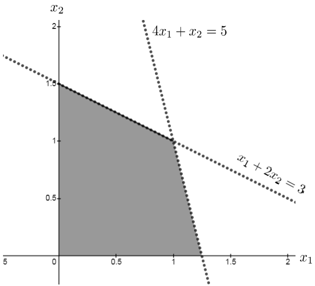
$$\begin{array}{rl} \text{minimize (or maximize)} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} \le \boldsymbol{b} \end{array}$$

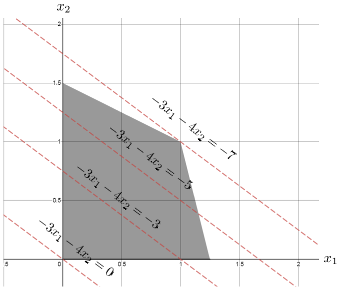


dashed lines are level sets $\boldsymbol{c}^T\boldsymbol{x} = \alpha$ for different α; the optimal solutions occur at an extreme point

Example 10.1

$$
\begin{array}{ll}
\text{minimize} & -3x_1 - 4x_2 \\
\text{subject to} & x_1 + 2x_2 \le 3 \\
& 4x_1 + x_2 \le 5 \\
& x_1 \ge 0, x_2 \ge 0
\end{array}
$$





optimal value is -7 achieved at unique $\boldsymbol{x}^\star = (1, 1)$

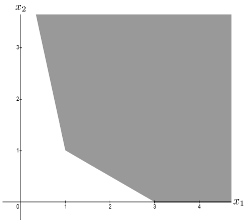
No feasible solution

$$\begin{array}{ll}\text{minimize} & -3x_1 - 4x_2 \\ \text{subject to} & x_1 + 2x_2 \geq 3 \\ & 4x_1 + x_2 \geq 5 \\ & x_1 + x_2 \leq -1 \\ & x_1 \geq 0, x_2 \geq 0\end{array}$$

- we added the additional constraint $x_1 + x_2 \leq -1$ to the previous problem
- there are no points that satisfy these constraints and we say that this is an *infeasible LP* and define the optimal value to be $+\infty$

No optimal solution

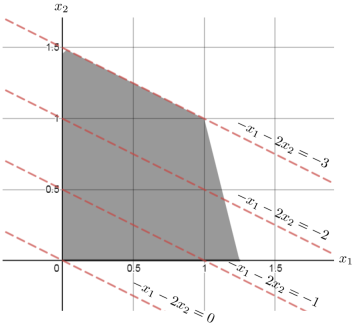
$$\begin{array}{ll} \text{minimize} & -3x_1 - 4x_2 \\ \text{subject to} & x_1 + 2x_2 \geq 3 \\ & 4x_1 + x_2 \geq 5 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$



- we can make the cost arbitrarily small by choosing x_1 and x_2 arbitrarily large; thus, there are no finite solutions to this problem
- the feasible set is said to be *unbounded* and the problem is said to be *unbounded below* and the optimal value is $-\infty$

Multiple optimal solutions

$$\begin{array}{ll} \text{minimize} & -x_1 - 2x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 3 \\ & 4x_1 + x_2 \leq 5 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$



- there are multiple points with minimum cost $\boldsymbol{c}^T\boldsymbol{x} = -3$
- any point on the line between the points $(0, 1.5)$ and $(1, 1)$ is optimal

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Standard form LP

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{10.5}$$

- any linear program can be transformed into the above standard form
- the constraints in a linear program may differ from one problem to another $A\boldsymbol{x} = \boldsymbol{b}$ and other problems have inequalities $A\boldsymbol{x} \leq \boldsymbol{b}$ and/or $\boldsymbol{x} \geq \boldsymbol{0}$
- to obtain systematic method for finding a solution, we need to transform any LP into a common form such as the standard form

Transformations: slack and surplus variables

- we can transform an inequality constraint into standard form as follows

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \iff \sum_{j=1}^{n} a_{ij} x_j + z_i = b_i, \quad z_i \geq 0$$

  where z_i is an additional variable, called *slack variable*

- similarly,

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \iff \sum_{j=1}^{n} a_{ij} x_j - z_i = b_i, \quad z_i \geq 0$$

  where z_i is called a *surplus variable*

Example 10.2

transform the following LP problem into standard form

$$\begin{array}{ll} \text{minimize} & c_1x_1 + c_2x_2 + c_3x_3 \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq b_2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array}$$

with variables x_1, x_2, x_3

we can introduce two variable x_4 and x_5, to transform the above problem into the following standard form:

$$\begin{array}{ll} \text{minimize} & c_1x_1 + c_2x_2 + c_3x_3 + 0x_4 + 0x_5 \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + x_4 = b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + x_5 = b_2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0 \end{array}$$

Transformations: free variables I

consider the problem

$$\begin{array}{ll} \text{minimize} & c_1x_1 + c_2x_2 \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 = b_1 \\ & a_{21}x_1 + a_{22}x_2 = b_2 \end{array}$$

where the constraint $\boldsymbol{x} \geq \mathbf{0}$ is not present

- we can rewrite that problem into standard from by replacing each entry x_i by $x_i = x_i^+ - x_i^-$ for $x_i^+ \geq 0$ and $x_i^- \geq 0$
- doing so, we get

$$\begin{array}{ll} \text{minimize} & c_1x_1^+ + c_2x_2^+ - c_1x_1^- - c_2x_2^- \\ \text{subject to} & a_{11}x_1^+ + a_{12}x_2^+ - a_{11}x_1^- - a_{12}x_2^- = b_1 \\ & a_{21}x_1^+ + a_{22}x_2^+ - a_{21}x_1^- - a_{22}x_2^- = b_2 \\ & x_1^+ \geq 0, x_2^+ \geq 0, x_1^- \geq 0, x_2^- \geq 0 \end{array}$$

  where we have four variables instead of two

Transformations: free variables II

in some situations, we can transform the problem into the standard by eliminating some variables; for example, consider

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 + x_3 \\ \text{subject to} & x_1 - x_2 - x_3 = 1 \\ & 2x_1 + x_2 + x_3 = 5 \\ & x_2 \geq 0, \quad x_3 \geq 0 \end{array}$$

- from the first equality we have $x_1 = 1 + x_2 + x_3$
- we can replace x_1 by $x_1 = 1 + x_2 + x_3$ in all other places to get

$$\begin{array}{ll} \text{minimize} & 2x_2 + 2x_3 + 1 \\ \text{subject to} & x_2 + x_3 = 1 \\ & x_2 \geq 0,\ x_3 \geq 0 \end{array}$$

and the constant 1 in the objective can be removed since it does not affect the solution

Example 10.3

transform the following LP into standard form

$$\begin{array}{ll} \text{maximize} & x_1 - x_2 \\ \text{subject to} & x_1 + x_2 \leq 1 \\ & x_1 + 2x_2 \geq 1 \\ & x_2 \geq 0 \end{array}$$

- we can represent this problem equivalently by

$$\begin{array}{ll} \text{minimize} & -x_1^+ + x_1^- + x_2 \\ \text{subject to} & x_1^+ - x_1^- + x_2 + z_1 = 1 \\ & x_1^+ - x_1^- + 2x_2 - z_2 = 1 \\ & x_1^+, x_1^-, x_2, z_1, z_2 \geq 0 \end{array}$$

- the objective is multiplied by a minus sign to transform the maximization into minimization

System of linear equations

consider the system of equations

$$A\boldsymbol{x} = \boldsymbol{b}$$

where A is an $m \times n$ matrix

- we assume that

the matrix A has linearly independent rows

- hence, the matrix A also has m linearly independent columns

- we assume that the columns of the matrix A are reordered such that the first m columns are linearly independent:

$$A = [B \; D]$$

where B is an $m \times m$ invertible matrix and D is an $m \times n - m$ matrix

Basic solutions

Basic solution the particular solution $\boldsymbol{x} = (B^{-1}\boldsymbol{b}, \mathbf{0})$ is called a *basic solution* to $A\boldsymbol{x} = \boldsymbol{b}$ with respect to the basis B

- the entries of $\boldsymbol{x}_B$ are called *basic variables* and the columns of B as *basic columns*
- if some of the variables of a basic solution are also zero, then the basic solution is said to be a *degenerate basic solution*

Basic feasible solution (BFS): a basic solution that is feasible for problem (10.5) ($\boldsymbol{x} \geq \mathbf{0}$) is called a *basic feasible solution*

- a *degenerate basic feasible solution* is a degenerate basic solution that is feasible
- if an optimal solution to (10.5) is also basic, then it is said to be an *optimal basic feasible solution*

Example 10.4

$$A = [\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3\ \boldsymbol{a}_4] = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

the matrix A has linearly independent rows

- if we choose $B = [\boldsymbol{a}_1\ \boldsymbol{a}_2]$, then $\boldsymbol{x}_B = B^{-1}\boldsymbol{b} = (6, 2)$; therefore, $\boldsymbol{x} = (6, 2, 0, 0)$ is a basic feasible solution with respect to the basis $B = [\boldsymbol{a}_1\ \boldsymbol{a}_2]$
- if we choose $B = [\boldsymbol{a}_3\ \boldsymbol{a}_4]$, then $\boldsymbol{x}_B = B^{-1}\boldsymbol{b} = (0, 2)$; thus, the point $\boldsymbol{x} = (0, 0, 0, 2)$ is a degenerate basic feasible solution with respect to $B = [\boldsymbol{a}_3\ \boldsymbol{a}_4]$
- if we choose $B = [\boldsymbol{a}_2\ \boldsymbol{a}_3]$, then $\boldsymbol{x}_B = B^{-1}\boldsymbol{b} = (2, -6)$; thus, the point $\boldsymbol{x} = (0, 2, -6, 0)$ is a basic solution with respect to $B = [\boldsymbol{a}_2\ \boldsymbol{a}_3]$, but is not feasible
- the point $\boldsymbol{x} = (3, 1, 0, 1)$ is a feasible solution that is not basic

Fundamental Theorem of LP

for a standard form LP, the *fundamental theorem of LP* states that

1. *if there exists a feasible solution, then there exists a basic feasible solution*
2. *if there exists an optimal feasible solution, then there exists an optimal basic feasible solution*

this means that to find an optimal solution, we only need to search the set of basic feasible solutions instead of looking at all possible solutions

Graphical interpretation

- the feasible set $\Omega = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ can be described as

$$\Omega = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\} = \{\boldsymbol{x} \mid \bar{A}\boldsymbol{x} \leq \bar{\boldsymbol{b}}\}$$

  where

$$\bar{A} = \begin{bmatrix} A \\ -A \\ -I \end{bmatrix}, \quad \bar{\boldsymbol{b}} = \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$$

- hence, the set Ω is a polyhedron

BFS and extreme points: the point $\hat{\boldsymbol{x}}$ is an extreme point of Ω if and only if, it is a basic feasible solution to $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$

References and further readings

- Stephen Boyd and Lieven Vandenberghe. *Convex Optimization,* Cambridge University Press, 2004, chapters 2.2.1, 2.2.4, 4.3.
- Edwin KP Chong and Stanislaw H Zak. *An Introduction to Optimization,* John Wiley & Sons, 2013, chapter 15.