7. The Laplace transform

- the Laplace transform
- properties of the Laplace transform
- solving differential equations
- circuit analysis using Laplace transform

The Laplace transform

the **Laplace transform** of $x(t)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \tag{7.1}$$

- variable s can be complex
- known as the *bilateral* or *two-sided* Laplace transform
- $x(t)$ is called the inverse Laplace transform of $X(s)$
- we use $x(t) \Longleftrightarrow X(s)$ to denote a Laplace transform pair

Region of convergence (ROC)

- the set of values of s for which the integral in Eq. (7.1) exists is called the *region of convergence* (ROC) for $X(s)$
- for a finite-duration, integrable signal $x_f(t)$, the ROC is the entire s-plane

Example 7.1

find the Laplace transform and the ROC for

(a) $x(t) = e^{-at}u(t)$

(b) $x(t) = -e^{-at}u(-t)$

Solution:

(a)

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-(s+a)t}\,dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_{0}^{\infty}$$

note that $t \longrightarrow \infty$, the term $e^{-(s+a)t}$ does not necessarily vanish because s is complex; here we recall that for a complex number $z = \alpha + j\beta$

$$e^{-zt} = e^{-(\alpha+j\beta)t} = e^{-\alpha t}e^{-j\beta t}$$

we have $\left|e^{-j\beta t}\right| = 1$ for any βt; therefore, as $t \longrightarrow \infty$, $e^{-zt} \longrightarrow 0$ only if $\alpha > 0$, and $e^{-zt} \longrightarrow \infty$ if $\alpha < 0$

we conclude that

$$\lim_{t\to\infty} e^{-(s+a)t} = \begin{cases} 0 & \operatorname{Re}(s+a) > 0 \\ \infty & \operatorname{Re}(s+a) < 0 \end{cases}$$

hence,

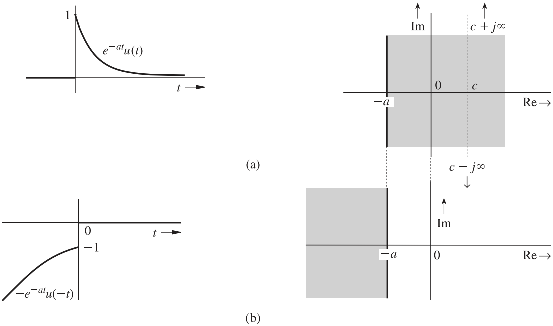
$$X(s) = \frac{1}{s+a} \quad \text{if} \quad \operatorname{Re} s > -a$$

the ROC is $\operatorname{Re} s > -a$

(b)

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} -e^{-at} u(-t) e^{-st}\, dt = -\int_{-\infty}^{0} e^{-(s+a)t}\, dt \\ &= \frac{1}{s+a} e^{-(s+a)t}\Big|_{-\infty}^{0} = \frac{1}{s+a} \quad \operatorname{Re} s < -a \end{aligned}$$

we see that $e^{-at}u(t)$ and $-e^{-at}u(-t)$ have identical $X(s)$ but different ROC



- for given $X(s)$, there may be more than one inverse transform, depending on the ROC; this increases the complexity in using the Laplace transform
- if we consider causal signals only, then there is a unique inverse transform of $X(s) = 1/(s+a)$, namely, $e^{-at}u(t)$ and there is no need to worry about ROC

Unilateral Laplace transform

the *unilateral Laplace transform* $X(s)$ of a signal $x(t)$ is

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt \tag{7.2}$$

- the unilateral transform is the bilateral transform that deals with a subclass of signals starting at $t = 0$ (causal signals)
- the 0^- in the lower limit means that even if $x(t)$ is discontinuous at $t = 0$, we can start the integration prior to the discontinuity as long as the integral converges (impulse function)
- the unilateral Laplace transform of any signal is *unique*, that is, for a given $X(s)$, there is a unique inverse transform $x(t)$

Existence

the unilateral Laplace transform exists if there exists a real σ such that:

$$\int_{0^-}^{\infty} \left| x(t) e^{-\sigma t} \right| dt < \infty \qquad (7.3)$$

- if $|x(t)| \leq M e^{\sigma_0 t}$ for some M and σ_0, then $X(s)$ exists for $\sigma > \sigma_0$
- e^{t^2} grows at a rate faster than $e^{\sigma_0 t}$; hence not Laplace-transformable

Abscissa of convergence: the smallest value of σ, denoted by σ_0, for which the integral in Eq. (7.3) is finite, is called the *abscissa of convergence*

- the ROC of $X(s)$ is $\mathrm{Re}\, s > \sigma_0$
- the abscissa of convergence for $e^{-at}u(t)$ is $-a$ (ROC is $\mathrm{Re}\, s > -a$)

Linearity

if

$$x_1(t) \Longleftrightarrow X_1(s) \quad \text{and} \quad x_2(t) \Longleftrightarrow X_2(s)$$

then

$$a_1 x_1(t) + a_2 x_2(t) \Longleftrightarrow a_1 X_1(s) + a_2 X_2(s)$$

Proof: by definition,

$$\begin{aligned}
\mathcal{L}\left[a_1 x_1(t) + a_2 x_2(t)\right] &= \int_{-\infty}^{\infty} \left[a_1 x_1(t) + a_2 x_2(t)\right] e^{-st}\, dt \\
&= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-st}\, dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-st}\, dt \\
&= a_1 X_1(s) + a_2 X_2(s)
\end{aligned}$$

Inverse Laplace transform

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} \, ds$$

- c is a constant chosen to ensure the convergence of the integral (7.1)
- the path of integration is along $c + j\omega$, with ω varying from $-\infty$ to ∞; moreover, the path of integration must lie in the ROC (or existence) for $X(s)$; for the signal $e^{-at}u(t)$, this is possible if $c > -a$; one possible path of integration is shown (dotted) in the figure on slide 7.4
- integration in the complex plane is beyond the scope of this course

Notation: the Laplace and inverse Laplace operations are denoted by:

$$X(s) = \mathcal{L}[x(t)] \quad \text{and} \quad x(t) = \mathcal{L}^{-1}[X(s)]$$

- note that

$$\mathcal{L}^{-1}\{\mathcal{L}[x(t)]\} = x(t) \quad \text{and} \quad \mathcal{L}\left\{\mathcal{L}^{-1}[X(s)]\right\} = X(s)$$

Common Laplace transform pairs

$x(t)$	$X(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$

(see Laplace table for more pairs)

Finding inverse Laplace

we can easily find the inverse transforms from Laplace tables if we can obtain a partial-fraction expansion of $X(s)$

Partial fraction expansion: if $X(s)$ is as a rational function, then

$$X(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^M + b_1 s^{M-1} + \cdots + b_{M-1} s + b_M}{(s-p_1)(s-p_2)\ldots(s-p_N)}$$

- values of s for which $X(s) = 0$ (*e.g.*, $P(s) = 0$) are the *zeros* of $X(s)$
- the values of s for which $X(s) \to \infty$ (*e.g.*, $Q(s) = 0$) are the *poles* of $X(s)$
- we can then further expand $X(s)$ using partial fraction expansion and find the inverse Laplace from the tables
- the ROC of unilateral transform is the region of the s-plane to the right of all the finite poles of the transform $X(s)$

Example 7.2

find the inverse unilateral Laplace transforms of

(a) $\dfrac{7s-6}{s^2-s-6}$ (real distinct roots)

(b) $\dfrac{2s^2+5}{s^2+3s+2}$ (improper $M = N$)

(c) $\dfrac{6(s+34)}{s\,(s^2+10s+34)}$ (complex distinct roots)

(d) $\dfrac{8s+10}{(s+1)(s+2)^3}$ (repeated roots)

Solution: we need to expand these functions into partial fractions

(a)

$$X(s) = \frac{7s-6}{(s+2)(s-3)} = \frac{k_1}{s+2} + \frac{k_2}{s-3}$$

we have

$$k_1 = \left.\frac{7s-6}{\cancel{(s+2)}(s-3)}\right|_{s=-2} = \frac{-14-6}{-2-3} = 4$$

$$k_2 = \left.\frac{7s-6}{(s+2)\cancel{(s-3)}}\right|_{s=3} = \frac{21-6}{3+2} = 3$$

therefore,

$$X(s) = \frac{7s-6}{(s+2)(s-3)} = \frac{4}{s+2} + \frac{3}{s-3}$$

using the Laplace table (pair 5), we have

$$x(t) = \mathcal{L}^{-1}\left(\frac{4}{s+2} + \frac{3}{s-3}\right) = \left(4e^{-2t} + 3e^{3t}\right)u(t)$$

(b) observe that $X(s)$ is an improper function with $M = N$; in this case, we can express $X(s)$ as:

$$X(s) = \frac{2s^2+5}{s^2+3s+2} = \frac{2s^2+5}{(s+1)(s+2)} = 2 + \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

where

$$k_1 = \left.\frac{2s^2+5}{\cancel{(s+1)}(s+2)}\right|_{s=-1} = \frac{2+5}{-1+2} = 7$$

$$k_2 = \left.\frac{2s^2+5}{\cancel{(s+1)}(s+2)}\right|_{s=-2} = \frac{8+5}{-2+1} = -13$$

therefore,

$$X(s) = 2 + \frac{7}{s+1} - \frac{13}{s+2}$$

from Laplace table (pair 2 and 5), we have

$$x(t) = 2\delta(t) + \left(7e^{-t} - 13e^{-2t}\right)u(t)$$

(c)

$$X(s) = \frac{6(s+34)}{s\left(s^2+10s+34\right)} = \frac{6(s+34)}{s(s+5-j3)(s+5+j3)}$$
$$= \frac{k_1}{s} + \frac{k_2}{s+5-j3} + \frac{k_2^*}{s+5+j3}$$

the coefficients (k_2 and k_2^*) of the conjugate terms must also be conjugate; we have $k_1 = 6$, and

$$k_2 = -3 + j4 = 5e^{j126.9^\circ}, \qquad k_2^* = 5e^{-j126.9^\circ}$$

hence

$$X(s) = \frac{6}{s} + \frac{5e^{j126.9^\circ}}{s+5-j3} + \frac{5e^{-j126.9^\circ}}{s+5+j3}$$

from Laplace table (pairs 2 and 10b), we obtain

$$x(t) = \left[6 + 10e^{-5t}\cos\left(3t + 126.9^\circ\right)\right]u(t)$$

(c) *Alternative approach:* to avoid dealing with complex numbers, we can express $X(s)$ as:

$$X(s) = \frac{6(s+34)}{s\,(s^2+10s+34)} = \frac{k_1}{s} + \frac{As+B}{s^2+10s+34}$$
$$= \frac{6}{s} + \frac{As+B}{s^2+10s+34}$$

where $k_1 = 6$ is already determined from before; to determine A we can multiply both sides by s and then let $s \to \infty$:

$$0 = 6 + A \implies A = -6$$

therefore,

$$\frac{6(s+34)}{s\,(s^2+10s+34)} = \frac{6}{s} + \frac{-6s+B}{s^2+10s+34}$$

to find B, we let s be any convenient value, say, $s = 1$, to obtain

$$\frac{210}{45} = 6 + \frac{B-6}{45} \Longrightarrow B = -54$$

and

$$X(s) = \frac{6}{s} + \frac{-6s - 54}{s^2 + 10s + 34}$$

using table (pairs 2 and 10c) with $A = -6$, $B = -54$, $a = 5$, $c = 34$, $b = \sqrt{c - a^2} = 3$, we have

$$r = \sqrt{\frac{A^2c + B^2 - 2ABa}{c - a^2}} = 10 \quad \theta = \tan^{-1} \frac{Aa - B}{A\sqrt{c - a^2}} = 126.9^\circ$$

therefore,

$$x(t) = \left[6 + 10e^{-5t} \cos\left(3t + 126.9^\circ\right)\right] u(t)$$

(d)

$$X(s)=\frac{8s+10}{(s+1)(s+2)^3}=\frac{k_1}{s+1}+\frac{a_0}{(s+2)^3}+\frac{a_1}{(s+2)^2}+\frac{a_2}{s+2}$$

where

$$k_1=\left.\frac{8s+10}{\cancel{(s+1)}(s+2)^3}\right|_{s=-1}=2$$

$$a_0=\left.\frac{8s+10}{(s+1)\cancel{(s+2)^3}}\right|_{s=-2}=6$$

$$a_1=\left\{\frac{d}{ds}\left[\frac{8s+10}{(s+1)\cancel{(s+2)^3}}\right]\right\}_{s=-2}=-2$$

$$a_2=\frac{1}{2}\left\{\frac{d^2}{ds^2}\left[\frac{8s+10}{(s+1)\cancel{(s+2)^3}}\right]\right\}_{s=-2}=-2$$

therefore,

$$X(s)=\frac{2}{s+1}+\frac{6}{(s+2)^3}-\frac{2}{(s+2)^2}-\frac{2}{s+2}$$

(d) *Alternative approach:* in this method, the simpler coefficients k_1 and a_0 are determined by the Heaviside "cover-up" procedure, as before; to determine the remaining coefficients, we use the clearing-fraction method:

$$\frac{8s+10}{(s+1)(s+2)^3} = \frac{2}{s+1} + \frac{6}{(s+2)^3} + \frac{a_1}{(s+2)^2} + \frac{a_2}{s+2}$$

if we multiply both sides by s and then let $s \to \infty$, we eliminate a_1:

$$0 = 2 + a_2 \Longrightarrow a_2 = -2$$

therefore,

$$\frac{8s+10}{(s+1)(s+2)^3} = \frac{2}{s+1} + \frac{6}{(s+2)^3} + \frac{a_1}{(s+2)^2} - \frac{2}{s+2}$$

a_1 can be determined by setting s equal to any convenient value, say, $s = 0$:

$$\tfrac{10}{8} = 2 + \tfrac{3}{4} + \tfrac{a_1}{4} - 1 \Longrightarrow a_1 = -2$$

therefore, $X(s) = \frac{2}{s+1} + \frac{6}{(s+2)^3} - \frac{2}{(s+2)^2} - \frac{2}{s+2}$, and from table, we have

$$x(t) = \left[2e^{-t} + \left(3t^2 - 2t - 2\right)e^{-2t}\right]u(t)$$

Example 7.3: improper

If $X_1(s) = P(s)/Q(s)$ is improper, where the order of $P(s)$ is greater than or equal to the order of $Q(s)$, then $P(s)$ must be divided by $Q(s)$ successively until the result has a remainder whose numerator is of order less than its denominator

Example

$$X_1(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

we must perform the indicated division until we obtain a remainder whose numerator is of order less than its denominator; hence,

$$X_1(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

Taking the inverse Laplace transform along with the differentiation theorem and the linearity theorem:

$$X_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathscr{L}^{-1}\left[\frac{2}{s^2 + s + 5}\right]$$

the inverse transform of $2/\left(s^2 + s + 5\right)$ can be found using partial-fraction expansion

Example 7.4

use the MATLAB `residue` command and Laplace table, to determine the inverse Laplace transform of each of the following functions:

(a) $X_a(s) = \dfrac{2s^2 + 5}{s^2 + 3s + 2}$

(b) $X_b(s) = \dfrac{2s^2 + 7s + 4}{(s + 1)(s + 2)^2}$

(c) $X_c(s) = \dfrac{8s^2 + 21s + 19}{(s + 2)\,(s^2 + s + 7)}$

Solution:

(a)
```
>> num = [2 0 5]; den = [1 3 2];
>> [r, p, k] = residue(num,den)
r = -13
7
p = -2
-1
k = 2
```
$X_a(s) = -13/(s+2) + 7/(s+1) + 2$ and
$x_a(t) = \left(-13e^{-2t} + 7e^{-t}\right) u(t) + 2\delta(t)$

(b)
```
>> num = [2 7 4]; den = [conv([1 1],conv([1 2],[1 2]))];
>> [r, p, k] = residue(num,den)
r = 3
2
-1
p = -2
-2
-1
k = []
```
$X_b(s) = 3/(s+2) + 2/(s+2)^2 - 1/(s+1)$ and
$x_b(t) = \left(3e^{-2t} + 2te^{-2t} - e^{-t}\right) u(t)$

(c)
```
>> num = [8 21 19]; den = [conv([1 2],[1 1 7])];
>> [r, p, k]= residue(num,den)
r = 3.5000-0.48113i
3.5000+0.48113i
1.0000
p = -0.5000+2.5981i
-0.5000-2.5981i
-2.0000
k = []
>> ang = angle(r), mag = abs(r)
ang = -0.13661
0.13661
0
mag = 3.5329
3.5329
1.0000
```

$$X_c(s) = \frac{1}{s+2} + \frac{3.5329e^{-j0.13661}}{s+0.5-j2.5981} + \frac{3.5329e^{j0.13661}}{s+0.5+j2.5981}$$

and

$$x_c(t) = \left[e^{-2t} + 1.7665e^{-0.5t}\cos(2.5981t - 0.1366)\right]u(t)$$

Finding the Laplace transform using Matlab

we can use MATLAB's symbolic math toolbox, determine the Laplace or inverse Laplace transform

Examples:

(a) the direct unilateral Laplace transform of $x_a(t) = \sin(at) + \cos(bt)$

```
>> syms a b t; x_a = sin(a*t)+cos(b*t);
>> X_a = laplace(x_a);
X_a = a/(a^2 + s^2) + s/(b^2 + s^2)
```

we express in standard rational form

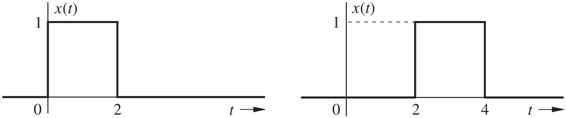
```
>> X_a = collect(X_a)
X_a = (a^2*s+a*b^2+a*s^2+s^3)/(s^4+(a^2 + b^2)*s^2+a^2*b^2)
```

(b) the inverse unilateral Laplace transform of $X_b(s) = as^2/\left(s^2 + b^2\right)$

```
>> syms a b s; X_b = (a*s^2)/(s^2+b^2);
>> x_b = ilaplace(X_b)
x_b = a*dirac(t) - a*b*sin(b*t)
```

Exercises

- by direct integration, find the Laplace transform and the ROC for $x(t)$



(a) (b)

Answer: (a) $\frac{1}{s}(1 - e^{-2s})$ for all s; (b) $\frac{1}{s}(1 - e^{-2s})e^{-2s}$ for all s

- use Laplace transform table to show that the Laplace transform of $10e^{-3t}\cos(4t + 53.13°)$ is $(6s - 14)/\left(s^2 + 6s + 25\right)$

- find the inverse Laplace transform of the following:

  (a) $\dfrac{s + 17}{s^2 + 4s - 5}$

  (b) $\dfrac{3s - 5}{(s + 1)\left(s^2 + 2s + 5\right)}$

  (c) $\dfrac{16s + 43}{(s - 2)(s + 3)^2}$

Answers:

(a) $(3e^{t} - 2e^{-5t})u(t)$ (b) $[-2e^{-t} + \frac{5}{2}e^{-t}\cos(2t - 36.87°)]u(t)$

(c) $[3e^{2t} + (t - 3)e^{-3t}]u(t)$

Outline

- the Laplace transform
- **properties of the Laplace transform**
- solving differential equations
- circuit analysis using Laplace transform

Shifting

Time-shifting: if $x(t) \Longleftrightarrow X(s)$ then for $t_0 \geq 0$

$$x\left(t-t_0\right) \Longleftrightarrow X(s) e^{-s t_0}$$

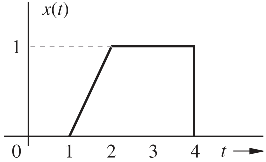
- here $x(t)$ is causal, and therefore, $x\left(t-t_0\right)$ starts at $t=t_0$ (we often avoid this ambiguity by considering $x(t) u(t)$)
- holds only for positive t_0 because if t_0 were negative, the signal $x\left(t-t_0\right)$ may not be causal

Frequency-shifting: if $x(t) \Longleftrightarrow X(s)$ then

$$x(t) e^{s_0 t} \Longleftrightarrow X\left(s-s_0\right)$$

Example 7.5

find the Laplace transform of $x(t)$ shown below



Solution: we can express the signal as:

$$\begin{aligned} x(t) &= (t-1)[u(t-1) - u(t-2)] + [u(t-2) - u(t-4)] \\ &= (t-1)u(t-1) - (t-1)u(t-2) + u(t-2) - u(t-4) \end{aligned}$$

we can rearrange the second term as

$$(t-1)u(t-2) = (t-2+1)u(t-2) = (t-2)u(t-2) + u(t-2)$$

hence,

$$x(t) = (t-1)u(t-1) - (t-2)u(t-2) - u(t-4)$$

application of the time-shifting property to $tu(t) \iff 1/s^2$ yields

$$(t-1)u(t-1) \iff \frac{1}{s^2}e^{-s} \quad \text{and} \quad (t-2)u(t-2) \iff \frac{1}{s^2}e^{-2s}$$

also

$$u(t) \iff \frac{1}{s} \quad \text{and} \quad u(t-4) \iff \frac{1}{s}e^{-4s}$$

therefore,

$$X(s) = \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s}$$

Example 7.6

find the inverse Laplace transform of

$$X(s) = \frac{s+3+5e^{-2s}}{(s+1)(s+2)}$$

Solution: we have

$$X(s) = \underbrace{\frac{s+3}{(s+1)(s+2)}}_{X_1(s)} + \underbrace{\frac{5e^{-2s}}{(s+1)(s+2)}}_{X_2(s)e^{-2s}}$$

where

$$X_1(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$X_2(s) = \frac{5}{(s+1)(s+2)} = \frac{5}{s+1} - \frac{5}{s+2}$$

therefore,

$$\begin{aligned} x_1(t) &= \left(2e^{-t} - e^{-2t}\right) u(t) \\ x_2(t) &= 5\left(e^{-t} - e^{-2t}\right) u(t) \end{aligned}$$

also, because

$$X(s) = X_1(s) + X_2(s)e^{-2s}$$

we can write

$$\begin{aligned} x(t) &= x_1(t) + x_2(t-2) \\ &= \left(2e^{-t} - e^{-2t}\right) u(t) + 5\left[e^{-(t-2)} - e^{-2(t-2)}\right] u(t-2) \end{aligned}$$

Differentiation

Time-differentiation: if $x(t) \Longleftrightarrow X(s)$ then

$$\frac{dx(t)}{dt} \Longleftrightarrow sX(s) - x\left(0^-\right)$$

repeated differentiation yields

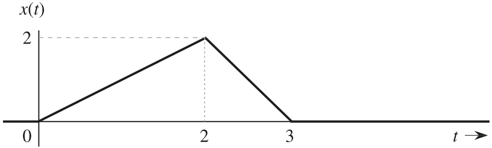
$$\begin{aligned}\frac{d^n x(t)}{dt^n} \Longleftrightarrow\ & s^n X(s) - s^{n-1} x\left(0^-\right) - s^{n-2} \dot{x}\left(0^-\right) - \cdots - x^{(n-1)}\left(0^-\right) \\ &= s^n X(s) - \sum_{k=1}^{n} s^{n-k} x^{(k-1)}\left(0^-\right)\end{aligned}$$

Frequency-differentiation: if $x(t) \Longleftrightarrow X(s)$ then

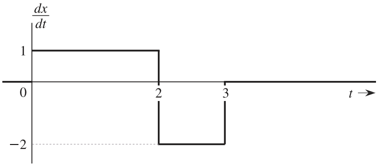
$$t^n x(t) \Longleftrightarrow (-1)^n \frac{d^n}{ds^n} X(s)$$

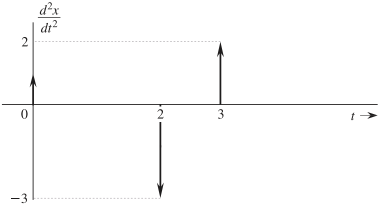
Example 7.7

find the Laplace transform of the signal $x(t)$ shown below by using Laplace table and the time-differentiation and time-shifting properties



Solution: the derivative at a point of jump discontinuity is an impulse of strength equal to the amount of jump





therefore,

$$\frac{d^2x(t)}{dt^2} = \delta(t) - 3\delta(t-2) + 2\delta(t-3)$$

the Laplace transform of this equation yields

$$\mathcal{L}\left(\frac{d^2x(t)}{dt^2}\right) = \mathcal{L}[\delta(t) - 3\delta(t-2) + 2\delta(t-3)]$$

using the time-differentiation property, the time-shifting property, and the facts that $x\left(0^-\right) = \dot{x}\left(0^-\right) = 0$, and $\delta(t) \Longleftrightarrow 1$, we obtain

$$s^2X(s) - 0 - 0 = 1 - 3e^{-2s} + 2e^{-3s}$$

thus,

$$X(s) = \frac{1}{s^2}\left(1 - 3e^{-2s} + 2e^{-3s}\right)$$

Integration

Time-integration: if $x(t) \Longleftrightarrow X(s)$ then

$$\int_{0^-}^{t} x(\tau)d\tau \Longleftrightarrow \frac{X(s)}{s}$$

and

$$\int_{-\infty}^{t} x(\tau)d\tau \Longleftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\tau)d\tau}{s}$$

Frequency-integration: if $x(t) \Longleftrightarrow X(s)$ then

$$\frac{x(t)}{t} \Longleftrightarrow \int_{s}^{\infty} X(u)du$$

Scaling and complex conjugation

the *scaling* property states that if $x(t) \Longleftrightarrow X(s)$, then for $a > 0$

$$x(at) \Longleftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$$

- time compression of a signal by a factor a causes expansion of its Laplace transform in the s scale by the same factor
- time expansion $x(t)$ causes compression of $X(s)$ by the same factor

Complex conjugation: if $x(t) \Longleftrightarrow X(s)$, then

$$x^*(t) \Longleftrightarrow X^*(s^*)$$

Convolution

let

$$x_1(t) \Longleftrightarrow X_1(s) \quad \text{and} \quad x_2(t) \Longleftrightarrow X_2(s)$$

Time-convolution

$$x_1(t) * x_2(t) \Longleftrightarrow X_1(s)X_2(s)$$

Frequency-convolution

$$x_1(t)x_2(t) \Longleftrightarrow \frac{1}{2\pi j}\left[X_1(s) * X_2(s)\right]$$

Example 7.8

use the time-convolution property of the Laplace transform to determine

$$c(t) = e^{at}u(t) * e^{bt}u(t)$$

Solution: using time-convolution property, we have

$$C(s) = \frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

the inverse transform of this equation yields

$$c(t) = \frac{1}{a-b}\left(e^{at} - e^{bt}\right)u(t)$$

Initial and final value theorems

Initial value theorem

$$x\left(0^{+}\right) = \lim_{s\to\infty} sX(s)$$

- applies only if $X(s)$ is strictly proper $(M < N)$
- for $M \geq N$, $\lim_{s\to\infty} sX(s)$ does not exist; in such a case, we must express $X(s)$ as a polynomial in s plus a strictly proper fraction, where $M < N$

Final value theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

- applies only if the poles of $X(s)$ are in the LHP (including $s = 0$)
- If there is a pole at the origin, then $x(t)$ contains a constant term, and hence, $x(\infty)$ exists and is a constant

Example 7.9

determine the initial and final values of $y(t)$ if

(a) $Y(s) = \dfrac{10(2s+3)}{s(s^2+2s+5)}$ (b) $Y(s) = \dfrac{s^3+3s^2+s+1}{s^2+2s+1}$

Solution:

(a) directly applying the theorems:

$$y(0^+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \frac{10(2s+3)}{(s^2+2s+5)} = 0$$

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{10(2s+3)}{(s^2+2s+5)} = 6$$

(b) here $M > N$, to use use the I.V.T, we write

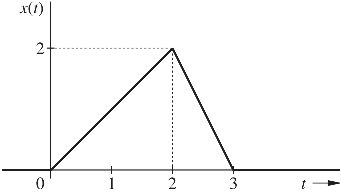
$$Y(s) = (s+1) - \frac{2s}{s^2+2s+1}$$

the inverse transform of $s+1$ is $\dot{\delta}(t) + \delta(t)$, which are zero at $t = 0^+$; hence:

$$y(0^+) = \lim_{s\to\infty} \frac{-2s^2}{s^2+2s+1} = -2, \qquad y(\infty) = \lim_{s\to 0} sY(s) = 0$$

Exercises

- find the Laplace transform of the signal illustrated below



Answer: $\frac{1}{s^2}(1 - 3e^{-2s} + 2e^{-3s})$

- find the inverse Laplace transform of $X(s) = \dfrac{3e^{-2s}}{(s-1)(s+2)}$

Answer: $(e^{t-2} - e^{-2(t-2)})u(t-2)$

Outline

- the Laplace transform
- properties of the Laplace transform
- **solving differential equations**
- circuit analysis using Laplace transform

Solving differential equations

- Laplace transform is a powerful tool to analyze of linear system dynamics
- using Laplace transform, the solution of the differential equation can be transformed into the solution of an algebraic equation
- using the Laplace transform we can solve differential equations knowing only initial conditions before the discontinuity 0^-
- directly solving using differential equations, we have to also know the initial conditions after the discontinuity 0^+

Example: use the Laplace transform to solve the second-order linear differential equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

with initial conditions $y\left(0^-\right) = 2$ and $\dot{y}\left(0^-\right) = 1$ and the input $x(t) = e^{-4t}u(t)$

Solution:

let $y(t) \Longleftrightarrow Y(s)$, then

$$\frac{dy(t)}{dt} \Longleftrightarrow sY(s) - y(0^-) = sY(s) - 2$$

$$\frac{d^2y(t)}{dt^2} \Longleftrightarrow s^2Y(s) - sy(0^-) - \dot{y}(0^-) = s^2Y(s) - 2s - 1$$

moreover, for $x(t) = e^{-4t}u(t)$

$$X(s) = \frac{1}{s+4} \quad \text{and} \quad \frac{dx(t)}{dt} \Longleftrightarrow sX(s) - x(0^-) = \frac{s}{s+4} - 0 = \frac{s}{s+4}$$

taking the Laplace transform of the diff. equation:

$$\left[s^2Y(s) - 2s - 1\right] + 5[sY(s) - 2] + 6Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$$

rearranging, we obtain

$$\left(s^2+5s+6\right)Y(s)-(2s+11)=\frac{s+1}{s+4}$$

therefore,

$$Y(s)=\frac{2s+11}{s^2+5s+6}+\frac{s+1}{(s^2+5s+6)(s+4)}=\frac{2s^2+20s+45}{(s+2)(s+3)(s+4)}$$

expanding the right-hand side into partial fractions:

$$Y(s)=\frac{13/2}{s+2}-\frac{3}{s+3}-\frac{3/2}{s+4}$$

taking inverse Laplace transform:

$$y(t)=\left(\frac{13}{2}e^{-2t}-3e^{-3t}-\frac{3}{2}e^{-4t}\right)u(t)$$

Zero-input and zero-state components

- the initial conditions term in the response give rise to the zero-input response
- the input term give rise to the zero-state response

Example: in the previous example, we have

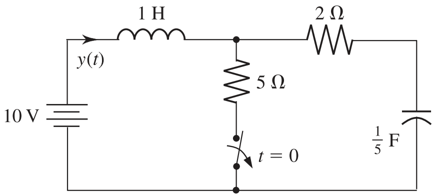
$$Y(s) = \underbrace{\frac{2s+11}{s^2+5s+6}}_{\text{initial conditions term}} + \underbrace{\frac{s+1}{(s+4)(s^2+5s+6)}}_{\text{input term}}$$

$$= \left[\frac{7}{s+2} - \frac{5}{s+3}\right] + \left[\frac{-1/2}{s+2} + \frac{2}{s+3} - \frac{3/2}{s+4}\right]$$

taking the inverse transform:

$$y(t) = \underbrace{\left(7e^{-2t} - 5e^{-3t}\right)u(t)}_{\text{ZIR}} + \underbrace{\left(-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t}\right)u(t)}_{\text{ZSR}}$$

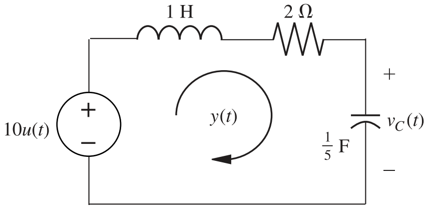
Example 7.10



the switch is in the closed position for a long time before $t = 0$, when it is opened instantaneously; find the inductor current $y(t)$ for $t \geq 0$

Solution: when the switch is in the closed position (for a long time), the inductor current is 2 amperes and the capacitor voltage is 10 volts; thus, $y(0^-) = 2$ and $v_C(0^-) = 10$

when the switch is opened, we get the circuit



notice that we represented the voltage source at $t \geq 0$ by the unit step $10u(t)$ for $t > 0$ after opening the switch

the loop equation of the circuit is

$$\frac{dy(t)}{dt} + 2y(t) + 5\int_{-\infty}^{t} y(\tau)d\tau = 10u(t)$$

if $y(t) \Longleftrightarrow Y(s)$ then

$$\frac{dy(t)}{dt} \Longleftrightarrow sY(s) - y\left(0^-\right) = sY(s) - 2$$

and

$$\int_{-\infty}^{t} y(\tau)d\tau \Longleftrightarrow \frac{Y(s)}{s} + \frac{\int_{-\infty}^{0^-} y(\tau)d\tau}{s}$$

note that $(1/C)\int_{-\infty}^{0^-} y(\tau)d\tau = v_C(0^-)$ and thus:

$$\int_{-\infty}^{0^-} y(\tau)d\tau = Cv_C\left(0^-\right) = \frac{1}{5}(10) = 2$$

hence

$$\int_{-\infty}^{t} y(\tau)d\tau \Longleftrightarrow \frac{Y(s)}{s} + \frac{2}{s}$$

using these results, the Laplace transform the diff. equation is

$$sY(s) - 2 + 2Y(s) + \frac{5Y(s)}{s} + \frac{10}{s} = \frac{10}{s}$$

thus

$$Y(s) = \frac{2s}{s^2 + 2s + 5}$$

to find the inverse Laplace transform of $Y(s)$, we use pair $10\mathrm{c}$ in Laplace table with values $A = 2$, $B = 0, a = 1$, and $c = 5$:

$$r = \sqrt{\frac{20}{4}} = \sqrt{5}, \quad b = \sqrt{c - a^2} = 2 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{2}{4}\right) = 26.6^\circ$$

therefore,

$$y(t) = \sqrt{5}e^{-t}\cos\left(2t + 26.6^\circ\right)u(t)$$

Exercise

- use the Laplace transform to solve the following differential equation for $y(t)$ if all initial conditions are zero

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

Answer: $y(t) = \left(1 - 2e^{-4t} + e^{-8t}\right) u(t)$

- use Laplace transform to solve

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + x(t)$$

for the input $x(t) = u(t)$; the initial conditions are $y(0^-) = 1$ and $\dot{y}(0^-) = 2$

Answer: $y(t) = \frac{1}{3}(1 + 9e^{-t} - 7e^{-3t})u(t)$

Outline

- the Laplace transform
- properties of the Laplace transform
- solving differential equations
- **circuit analysis using Laplace transform**

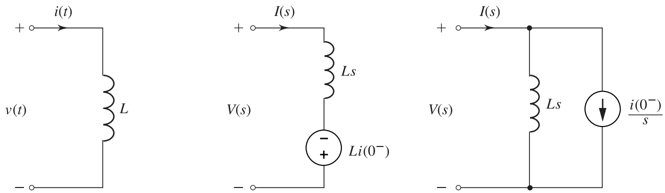
Laplace representation of basic electric elements

Resistor

$$v(t) = Ri(t) \iff V(s) = RI(s)$$

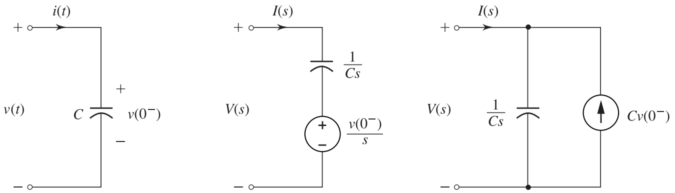
Inductor

$$v(t) = L\frac{di(t)}{dt} \iff V(s) = LsI(s) - Li\left(0^-\right)$$



Capacitor

$$i(t) = C\frac{dv(t)}{dt} \iff V(s) = \frac{1}{Cs}I(s) + \frac{v(0^-)}{s}$$



Impedance: the *impedance* of an element is $Z = V(s)/I(s)$ for the element (under zero initial conditions)

- the impedance of a resistor of R is $Z = R$
- the impedance of an inductor of L is $Z = Ls$
- the impedance of a capacitor C is $Z = 1/Cs$

Kirchhoff's laws

Time domain

$$\sum_{k=1}^{N} v_k(t) = 0 \quad \text{and} \quad \sum_{k=1}^{M} i_k(t) = 0$$

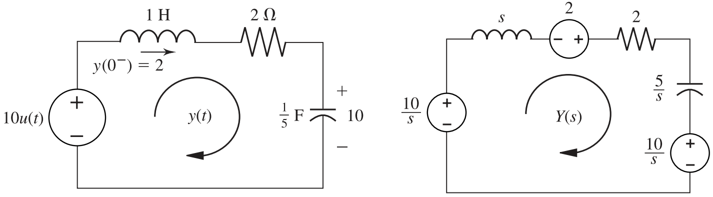
- $v_k(t)(k = 1, 2, \ldots, N)$ are the voltages across N elements in a loop
- $i_k(t)(k = 1, 2, \ldots, M)$ are the M currents entering a node

Laplace domain

$$\sum_{k=1}^{N} V_k(s) = 0 \quad \text{and} \quad \sum_{k=1}^{M} I_k(s) = 0$$

- $V_k(s)$ and $V_k(s)$ are the Laplace transforms of $v_k(t)$ and $i_k(t)$
- we can treat the network as if it consisted of the "resistances" R, Ls, $1/Cs$

Illustrative example



(a) (b)

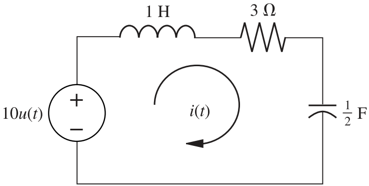
the initial conditions $y\left(0^-\right) = 2$ and $v_C\left(0^-\right) = 10$

the total voltage in the loop is $(10/s) + 2 - (10/s) = 2$, and the loop impedance is $(s + 2 + (5/s))$; therefore,

$$Y(s) = \frac{2}{s + 2 + 5/s} = \frac{2s}{s^2 + 2s + 5}$$

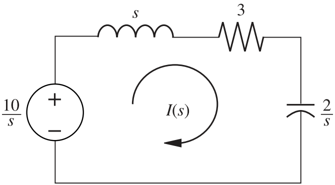
which matches our earlier result in slide 7.48

Example 7.11



find the loop current $i(t)$ in the circuit shown if all the initial conditions are zero

Solution: we first, we represent the circuit in the frequency domain:



total impedance in the loop is

$$Z(s) = s + 3 + \frac{2}{s} = \frac{s^2 + 3s + 2}{s}$$

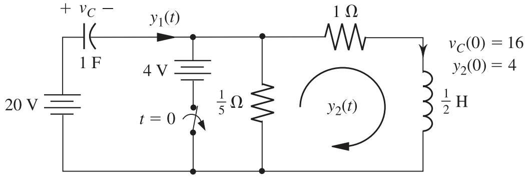
the input voltage is $V(s) = 10/s$; therefore:

$$\begin{aligned} I(s) = \frac{V(s)}{Z(s)} &= \frac{10}{s^2 + 3s + 2} \\ &= \frac{10}{(s+1)(s+2)} \\ &= \frac{10}{s+1} - \frac{10}{s+2} \end{aligned}$$

taking the inverse transform, we arrive at

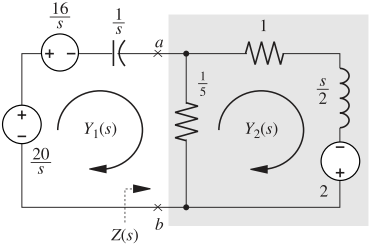
$$i(t) = 10\left(e^{-t} - e^{-2t}\right) u(t)$$

Example 7.12



the switch in the circuit is in the closed position for a long time before $t = 0$, when it is opened instantaneously; find the currents $y_1(t)$ and $y_2(t)$ for $t \geq 0$

Solution: by inspection, the initial conditions are $v_C\left(0^-\right) = 16$ and $y_2\left(0^-\right) = 4$; thus for $t \geq 0$, the circuit in Laplace domain is



the loop equations can be written directly in the frequency domain as

$$\frac{Y_1(s)}{s} + \frac{1}{5}\left[Y_1(s) - Y_2(s)\right] = \frac{4}{s}$$

$$-\frac{1}{5}Y_1(s) + \frac{6}{5}Y_2(s) + \frac{s}{2}Y_2(s) = 2$$

solving, we get

$$Y_1(s) = \frac{24(s+2)}{s^2+7s+12} = \frac{24(s+2)}{(s+3)(s+4)} = \frac{-24}{s+3} + \frac{48}{s+4}$$

and

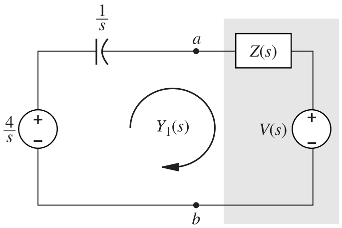
$$Y_2(s) = \frac{4(s+7)}{s^2+7s+12} = \frac{16}{s+3} - \frac{12}{s+4}$$

hence,

$$\begin{aligned} y_1(t) &= \left(-24e^{-3t} + 48e^{-4t}\right) u(t) \\ y_2(t) &= \left(16e^{-3t} - 12e^{-4t}\right) u(t) \end{aligned}$$

Alternative solution: we can also use Thévenin's theorem to compute $Y_1(s)$ and $Y_2(s)$; the previous circuit shows that the Thévenin impedance $Z(s)$ and the Thévenin source $V(s)$ (across right part of terminals ab) are

$$Z(s) = \frac{\frac{1}{5}\left(\frac{s}{2}+1\right)}{\frac{1}{5}+\frac{s}{2}+1} = \frac{s+2}{5s+12}, \qquad V(s) = \frac{-\frac{1}{5}}{\frac{1}{5}+\frac{s}{2}+1}2 = \frac{-4}{5s+12}$$

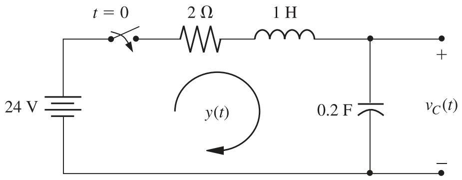


the current $Y_1(s)$ is given by

$$Y_1(s) = \frac{\frac{4}{s} - V(s)}{\frac{1}{s} + Z(s)} = \frac{24(s+2)}{s^2+7s+12}$$

which matches our previous result (we can determine $Y_2(s)$ in a similar manner)

Exercise



the input is switched on at $t = 0$; the initial conditions are $y(0^-) = 2$ amperes and $v_C(0^-) = 50$ volts

find the loop current $y(t)$ and the capacitor voltage $v_C(t)$ for $t \geq 0$

Answer:

$$y(t) = 10\sqrt{2}e^{-t} \cos(2t + 81.8°)\, u(t)$$

$$v_C(t) = \left[24 + 31.62e^{-t} \cos(2t - 34.7°)\right] u(t)$$

References

Reference:

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press, chapter 4 (4.1-4.4).