4. Discrete-time signals

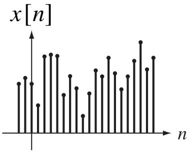
- DT signals
- signal operations
- useful DT signals
- signal energy and power
- aliasing and DT sinusoids

Discrete-time signals

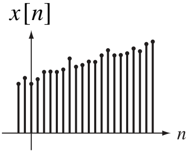
a *discrete-time (DT) signal* is a function defined over an *integer* variable

$$x[n] \quad \text{where} \quad n \in \{\ldots, -1, 0, 1, \ldots\}$$

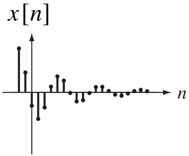
- a sequence of numbers $\ldots, x[-1], x[0], x[1], \ldots$
- a CT signal $x(t)$ can be transformed into a DT signal by sampling it $x[n] = x(t_n)$ over discrete instants $\{t_n\}$, $n = 0, 1, 2, \ldots$
- examples:



stock market daily averages



weekly average tempratures



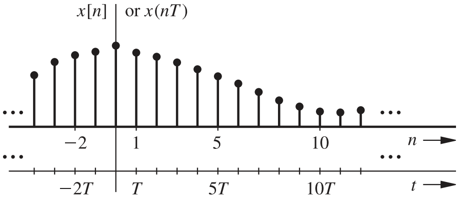
samples from exponentially damped sinusoid

Uniform sampling

uniform sampling a continuous-time signal $x(t)$ gives a DT signal:

$$x[n] = x(nT)$$

- n is an integer
- T is sampling period or sampling interval



Example: sampling $x(t) = e^{-t}$ with $T = 0.1$:

$$x[n] = e^{-nT} = e^{-0.1n} \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots$$

Causal and periodic signals

a signal $x[n]$ is *causal* if

$$x[n] = 0, \quad n < 0$$

- a signal $x[n]$ is *anticausal* if $x[n] = 0,\ n \geq 0$
- a signal that starts before $n = 0$ is called *noncausal*

Periodic signals: a signal $x[n]$ is *periodic* if for some positive constant N:

$$x[n] = x[n + N], \quad \text{for all } n$$

- *fundamental period* N_0 is the minimum N, such that the above holds
- fundamental frequency is $F_0 = 1/N_0$ cycles/sample and $\Omega_0 = 2\pi/N_0$ radians/sample
- a periodic signal must start at $n = -\infty$ and continue forever

Discrete-time sinusoid

$$A\cos(\Omega n + \theta) = A\cos(2\pi F n + \theta)$$

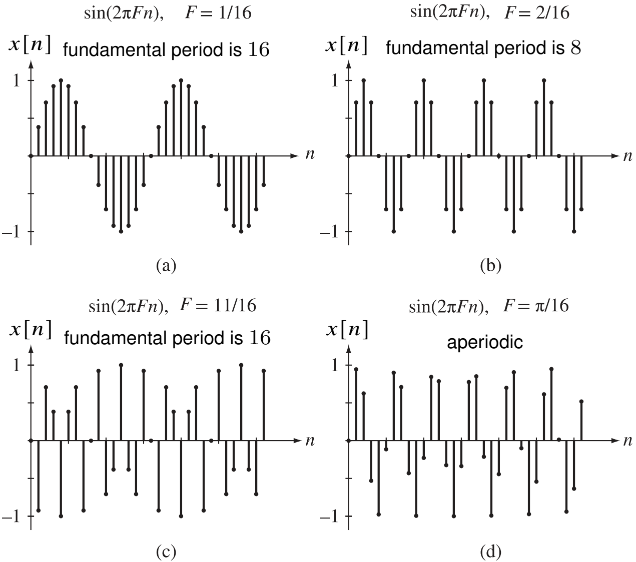
- A is the *amplitude*, θ is the *phase* in radians
- the *frequency* Ω has dimension *radians per sample*
- $F = \Omega/2\pi$ with dimension *cycles (radians/2π) per sample*
- uniform sampling of $x(t) = \cos\omega t$ with sampling rate T seconds gives

$$x[n] = \cos(\omega n T) = \cos(\Omega n) \quad \text{where} \quad \Omega = \omega T$$

Periodicity of DT sinusoid: the DT sinusoid $x[n] = \cos(\Omega n) = \cos(2\pi F n)$ is periodic if $\Omega N = 2\pi m$ for some non-zero integers m and N

- implies DT sinusoid is periodic if $F = m/N$ is a rational number
- if $F = m_0/N_0$ expressed in simplest form, then N_0 is the *fundamental period* in samples/cycle

Examples



$\sin(2\pi F n), \quad F = 1/16$
$x[n]$
fundamental period is 16
1
–1
n
(a)
$\sin(2\pi F n), \quad F = 2/16$
$x[n]$
fundamental period is 8
1
–1
n
(b)
$\sin(2\pi F n), \quad F = 11/16$
$x[n]$
fundamental period is 16
1
–1
n
(c)
$\sin(2\pi F n), \quad F = \pi/16$
$x[n]$
aperiodic
1
–1
n
(d)

Sum periodic signals

the sum of periodic DT signals is always periodic

- let $x_1[n]$ and $x_2[n]$ be periodic with fundamental periods N_{01} and N_{02}

$$x[n] = x_1[n] + x_2[n]$$

- $x[n]$ is periodic and the period is the least common multiple of N_{01} and N_{02}
- if $N_{01}/N_{02} = p/q$ for some integers p and q in smallest form, then $N_0 = \text{LCM}(N_{01}, N_{02}) = qN_{01} = pN_{02}$ is the fundamental period of $x[n]$

Example:

$$x[n] = 2\cos(9\pi n/4) - 3\sin(6\pi n/5)$$

we can write the function as

$$x[n] = 2\cos(2\pi(9/8)n) - 3\sin(2\pi(3/5)n)$$

we have $N_{01} = 8$ and $N_{02} = 5$; hence $x[n]$ is periodic and $N_0 = \text{LCM}(8, 5) = 40$

Discrete-time exponential

the *discrete-time exponential function* is

$$x[n] = \gamma^n$$

- can be expressed in usual form $\gamma^n = e^{\lambda n}$ where $\gamma = e^{\lambda}$
- for discrete-time signals, γ^n is preferred over $e^{\lambda n}$

Complex exponential: for complex $\gamma = re^{j\Omega}$, we get

$$x[n] = r^n e^{j\Omega n} = r^n(\cos \Omega n + j \sin \Omega n)$$

- the frequency is $|\Omega|$
- the angle is $n\Omega$
- in complex plane, $e^{j\Omega n}$ is a point on a unit circle at an angle Ωn

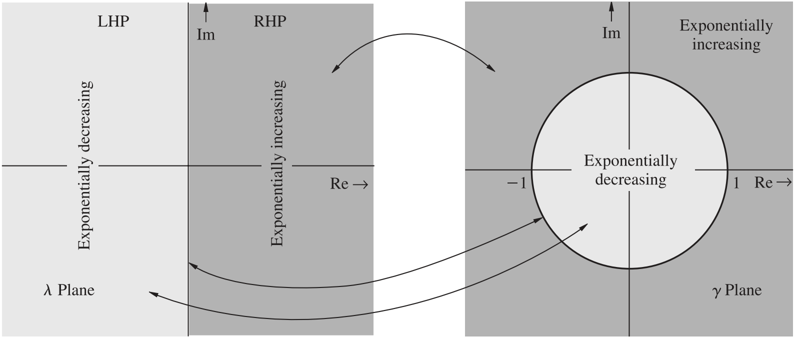
Nature of γ^n

Nature of $e^{\lambda n}$

- $e^{\lambda n}$ grows exponentially with n if $\mathrm{Re}\,\lambda > 0$ (λ in RHP)
- $e^{\lambda n}$ decays exponentially with n if $\mathrm{Re}\,\lambda < 0$ (λ in LHP)
- $e^{\lambda n}$ constant or oscillate if $\mathrm{Re}\,\lambda = 0$ (λ on imaginary axis)

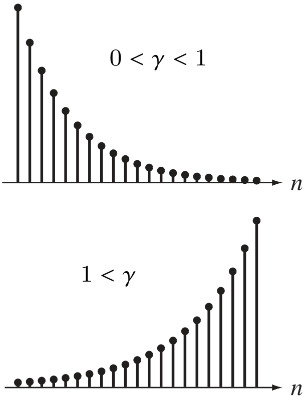
Nature of γ^n

- γ^n grows exponentially with n if $|\gamma| > 1$ (γ outside unit circle)
- γ^n decays exponentially with n if $|\gamma| < 1$ (γ inside unit circle)
- γ^n is a constant or oscillate if $|\gamma| = 1$ (γ on unit circle)

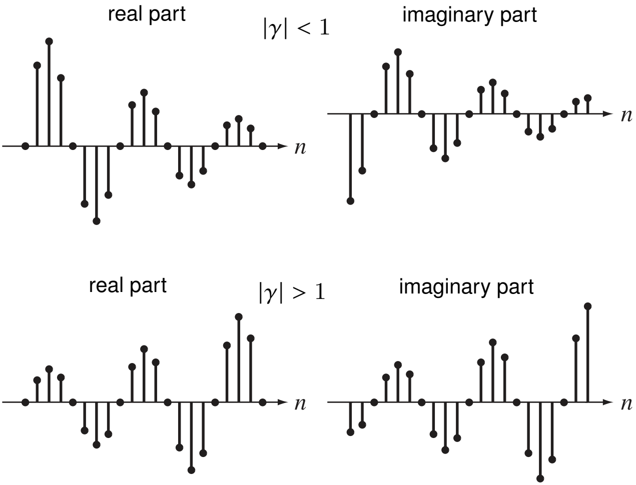


- for $\lambda = a + jb$, we have $e^{\lambda n} = \gamma^n$ where $\gamma = e^{\lambda} = e^a e^{jb}$
- hence, $|\gamma| = |e^a||e^{jb}| = e^a$

Behavior of γ^n for real γ



Behavior of γ^n for complex γ



Plotting DT signals in Matlab

we can use Matlab to discrete-time signals

Example: the code below plots the following signals over ($0 \leq n \leq 8$)

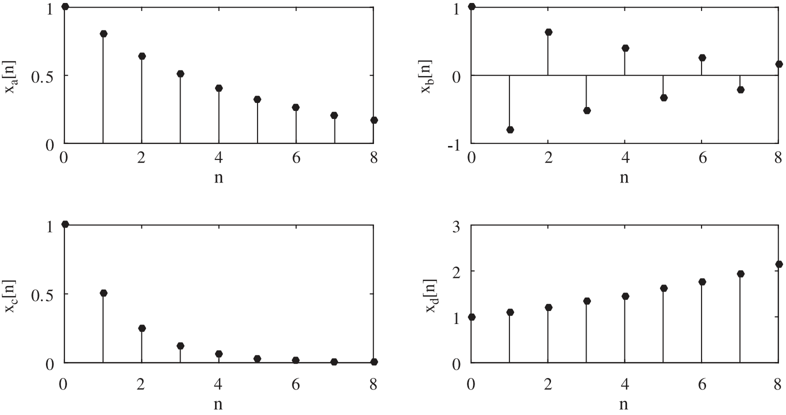
(a) $x_a[n] = (0.8)^n$

(b) $x_b[n] = (-0.8)^n$

(c) $x_c[n] = (0.5)^n$

(d) $x_d[n] = (1.1)^n$

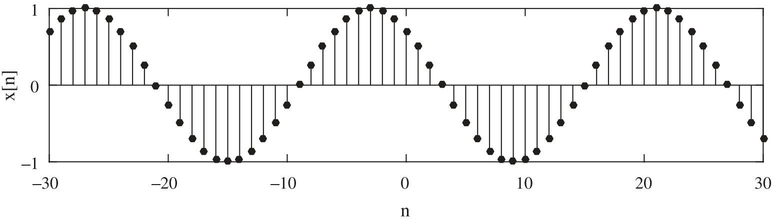
```
n = (0:8); x_a = @(n) (0.8).^n; x_b = @(n) (-0.8).^(n);
x_c = @(n) (0.5).^n; x_d = @(n) (1.1).^n;
subplot(2,2,1); stem(n,x_a(n),'k'); ylabel('x_a[n]'); xlabel('n');
    subplot(2,2,2); stem(n,x_b(n),'k'); ylabel('x_b[n]'); xlabel('n');
subplot(2,2,3); stem(n,x_c(n),'k'); ylabel('x_c[n]'); xlabel('n');
subplot(2,2,4); stem(n,x_d(n),'k'); ylabel('x_d[n]'); xlabel('n');
```



$x_a[n]$
1
0.5
0
0
2
4
6
8
n
$x_b[n]$
1
0
-1
0
2
4
6
8
n
$x_c[n]$
1
0.5
0
0
2
4
6
8
n
$x_d[n]$
3
2
1
0
0
2
4
6
8
n

Example: the code below plots $\cos(\frac{\pi}{12}n + \frac{\pi}{4})$ over the range $-30 \leq n \leq 30$

```
n = (-30:30); x = @(n) cos(n*pi/12+pi/4);
clf; stem(n,x(n),'k'); ylabel('x[n]'); xlabel('n');
```



Exercises

- show that the following equalities holds

  (a) $(0.25)^{-n} = 4^n$
  (b) $4^{-n} = (0.25)^n$
  (c) $e^{3n} = (20.086)^n$
  (d) $2^n = e^{0.693n}$
  (e) $e^{-1.5n} = (0.2231)^n = (4.4817)^{-n}$
  (f) $(0.5)^n = e^{-0.693n}$
  (g) $(0.8)^{-n} = e^{0.2231n}$

- determine and sketch the DT exponentials γ^n that result from sampling ($T = 1$) the following CT exponentials:

  (a) $e^{0t} = 1$
  (b) e^t
  (c) $e^{-0.6931t}$
  (d) $(-e^{-0.6931t})^t$
  (e) 2^t
  (f) 2^{-t}
  (g) $e^{-t/4}$
  (h) $e^{j\pi t}$

  for each case, locate the value γ in the complex plane and state whether γ^n is exponentially decaying, exponentially growing, or non-decaying

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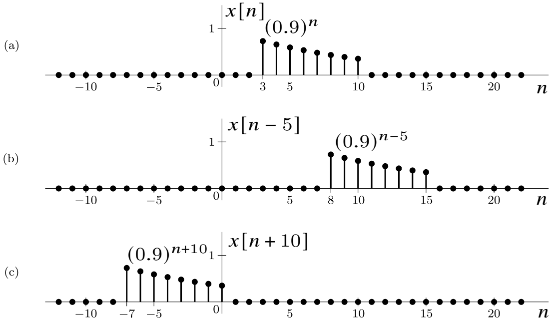
Time-shifting

the signal $x[n]$ can be shifted to the right or left by $n_0 > 0$ units

$$x[n - n_0] \qquad \text{(right-shifted (delayed) signal)}$$

$$x[n + n_0] \qquad \text{(left-shifted (advanced) signal)}$$

Example:

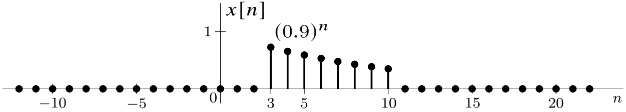


Time reversal

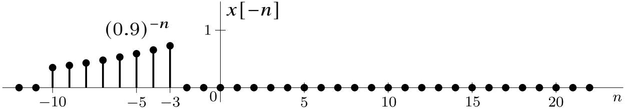
the *time reversal* operation $x[-n]$ rotates $x[n]$ about the vertical axis

Example:

(a)



(b)

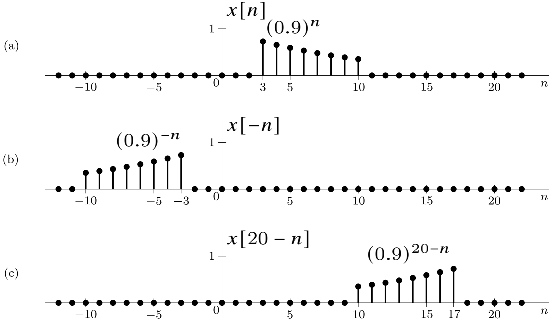


Time-reversal and shifting

the time-reversal and shifting operation is $x[k-n]$

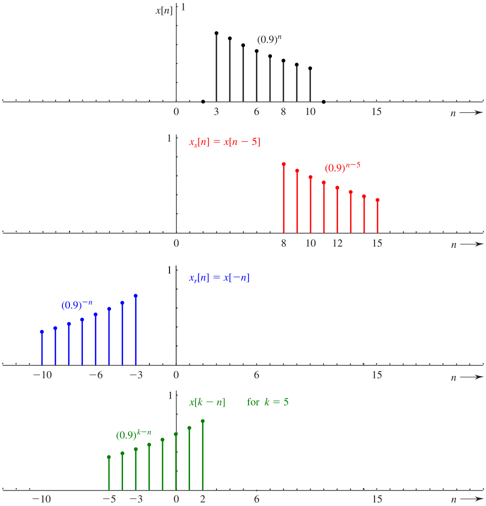
1. $x[n] \xrightarrow{\text{time reverse}} x[-n] \xrightarrow{\text{(right) shift by } k} x[-(n-k)] = x[k-n]$
2. $x[n] \xrightarrow{\text{(left) shift by } k} x[n+k] \xrightarrow{\text{time reverse}} x[k-n]$

Example: find $x[2-n]$



Example 4.1

plot $x[5-n]$ for the signal $x[n]$

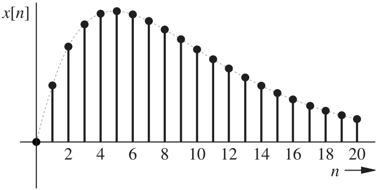


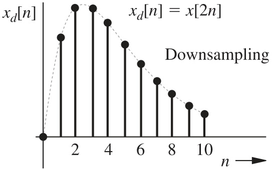
Downsampling (time compression)

downsampling is the compression of $x[n]$ by integer factor M:

$$x_d[n] = x[Mn]$$

- $x[Mn]$ selects every Mth sample of $x[n]$ ($x[0], x[M], x[2M], \ldots$)
- reduces the number of samples by factor M (loss of samples)
- if $x[n]$ is obtained by sampling a continuous-time signal, this operation implies reducing the sampling rate by factor M





Upsampling and interpolation

upsampling is the expansion of $x[n]$ by integer factor L

$$x_e[n] = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{otherwise } (n/L \text{ noninteger}) \end{cases}$$

- in general, for $n = 0, 1, 2, \ldots$, $x_e[n]$ is:

$$x[0], \underbrace{0, 0, \ldots, 0,}_{L-1 \text{ zeros}} x[1], \underbrace{0, 0, \ldots, 0,}_{L-1 \text{ zeros}} x[2], \underbrace{0, 0, \ldots, 0,}_{L-1 \text{ zeros}} x[3], \ldots$$

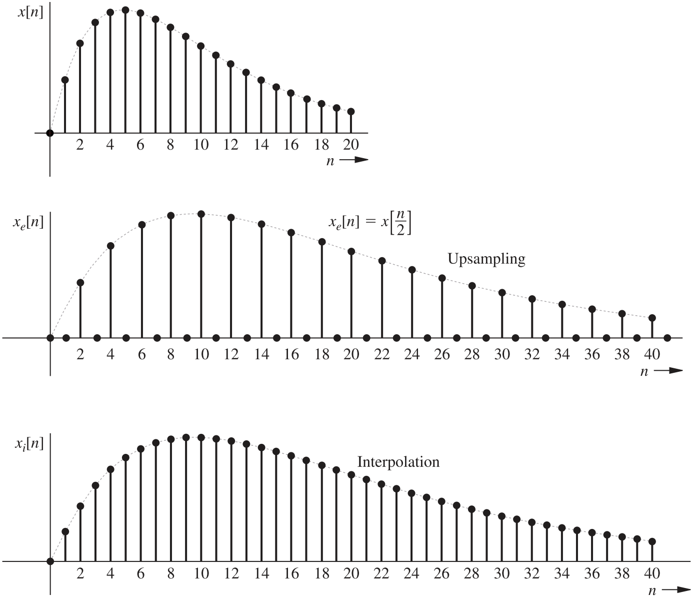
- the sampling rate of $x_e[n]$ is L times that of $x[n]$

Interpolation

- the process of filling-in the zero-valued samples is called *interpolation*
- for example, using *linear interpolation* for $L = 2$, the zero (odd-numbered) samples are replaced by:

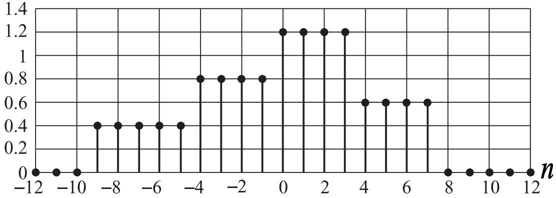
$$x_i[n] = \tfrac{1}{2}(x_e[n-1] + x_e[n+1])$$

Example


$x[n]$
2 4 6 8 10 12 14 16 18 20
n
$x_e[n]$
$x_e[n] = x\left[\frac{n}{2}\right]$
Upsampling
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
n
$x_i[n]$
Interpolation
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
n

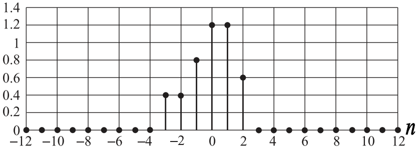
Example 4.2

sketch $x[-15-3n]$ for the DT signal show below

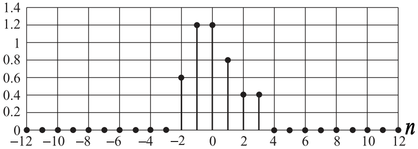


Solution: we write $x[-15-3n] = x[-3(n+5)]$ and then follows the steps given next

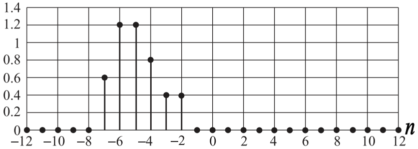
- compress $x[n]$ by 3 to get $x[3n]$



- time-reverse $x[3n]$ to get $x[-3n]$

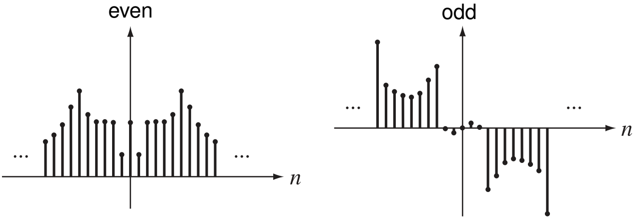


- left-shift $x[-3n]$ by 5 to obtain $x[-3(n+5)] = x[-15-3n]$



Even and odd signals

- a function $x_e[n]$ is **even** if $x_e[n] = x_e[-n]$
- a function $x_o[n]$ is **odd** if $x_o[n] = -x_o[-n]$



every signal $x[n]$ can expressed as

$$x[n] = \underbrace{\frac{1}{2}[x[n] + x[-n]]}_{\text{even}} + \underbrace{\frac{1}{2}[x[n] - x[-n]]}_{\text{odd}}$$

Example 4.3

find the even and odd parts of the function, $x[n] = \sin(2\pi n/7)\left(1 + n^2\right)$

Solution: the even part is

$$x_e[n] = \frac{\sin(2\pi n/7)\left(1 + n^2\right) + \sin(-2\pi n/7)\left(1 + (-n)^2\right)}{2} = 0$$

the odd part is

$$\begin{aligned} x_o[n] &= \frac{\sin(2\pi n/7)\left(1 + n^2\right) - \sin(-2\pi n/7)\left(1 + (-n)^2\right)}{2} \\ &= \sin(2\pi n/7)(1 + n^2) \end{aligned}$$

the function is odd since the even part is zero

Properties

Multiplications

$$\text{even function} \times \text{odd function} = \text{odd function}$$
$$\text{odd function} \times \text{odd function} = \text{even function}$$
$$\text{even function} \times \text{even function} = \text{even function}$$

Symmetric summation of even function: for positive integer N

$$\sum_{n=-N}^{N} x[n] = x[0] + 2\sum_{n=1}^{N} x[n] \qquad (x[n] \text{ is even})$$

Symmetric summation of odd function: for positive integer N

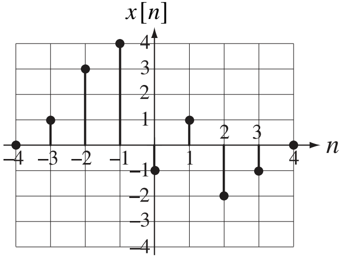
$$\sum_{n=-N}^{N} x[n] = 0 \qquad (x[n] \text{ is odd})$$

Exercises

- show that $x[n] = (0.9)^n$ for $3 \leq n \leq 10$ and zero otherwise left-shifted by 3 units can be expressed as $0.729(0.9)^n$ for $0 \leq n \leq 7$, and zero otherwise; sketch the shifted signal
- show that $x[-k-n]$ can be obtained from $x[n]$ by first right-shifting $x[n]$ by k units and then time-reversing this shifted signal
- sketch the signal $x[n] = e^{-0.5n}$ for $-3 \leq n \leq 2$, and zero otherwise; sketch the corresponding time-reversed signal and show that it can be expressed as $x_r[n] = e^{0.5n}$ for $-2 \leq n \leq 3$

Exercises

- consider the signal $x[n] = \cos(2\pi n/4)$, which are samples of the signal $\cos(2\pi t)$ taken at sampling rate $T = 1/4$; sketch $x[n]$, $x[2n]$, and $x[4n]$; comment on the results
- the operation $x[2n]$ represents a compression by 2 that preserves the even-numbered samples of the original signal $x[n]$; show that $x[2n+1]$ also compresses $x[n]$ by a factor of 2, but preserves the odd-numbered samples
- plot $y[n] = x[3n-1]$ for the signal $x[n]$ shown below



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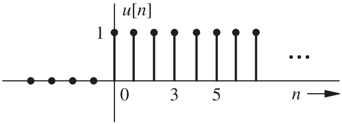
Unit step and unit ramp

(discrete-time) unit-step

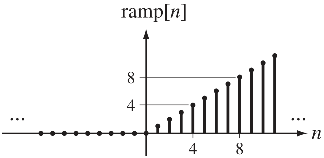
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0, & n < 0 \end{cases}$$



also called *unit-step sequence*

(discrete-time) unit-ramp

$$\text{ramp}[n] = \begin{cases} n & n > 0 \\ 0, & n \leq 0 \end{cases} = nu[n]$$

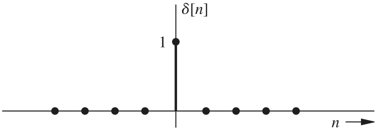


also called *unit-ramp sequence*

Unit impulse

(discrete-time) unit-impulse

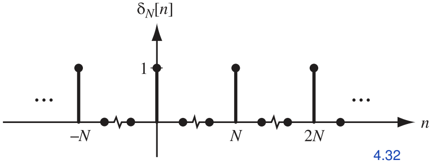
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



- also called *unit sample function* or *Kronecker delta function*
- defined everywhere (unlike continuous case)
- $\delta[n] = \delta[an]$ for any integer $a \neq 0$

unit periodic impulse (impulse train)

$$\delta_N[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$$



Properties

- *multiplication by DT impulse:*

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

- *sampling or sifting property:*

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Relation between unit step and unit impulse

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$\delta[n] = u[n] - u[n-1]$$

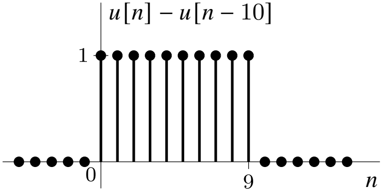
Rectangular sequence

the function

$$u[n - n_1] - u[n - n_2]$$

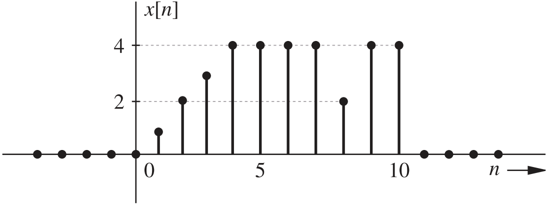
with $n_1 < n_2$ is a rectangular sequence from n_1 until $(n_2 - 1)$

Example:



Example 4.4

describe the signal $x[n]$ by a single expression valid for all n using unit-sequence



Solution: there are many ways to describe the signal using different but equivalent expressions; one expression is

$$x[n] = n(u[n] - u[n-5]) + 4(u[n-5] - u[n-11]) - 2\delta[n-8]$$

Outline

- DT signals
- signal operations
- useful DT signals
- **signal energy and power**
- aliasing and DT sinusoids

Energy and power signals

Energy of signal

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- finite if $|x[n]| \to 0$ as $|n| \to \infty$; infinite otherwise
- if E_x is finite, the signal is called an *energy signal*

Power of a signal

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- P_x is the time average (mean) of $|x[n]|^2$, also called *average power*
- $\sqrt{P_x}$ is the *rms* (root-mean-square) value of $x(t)$
- if P_x is finite and nonzero, the signal is called a *power signal*

- an energy signal has zero power
- a power signal has infinite energy
- hence, a signal cannot be both an energy signal and a power signal
- some signals are neither energy nor power signals

Periodic signals power: a periodic signal $x[n]$ with period N_0 has power

$$P_x = \frac{1}{N_0} \sum_{N_0} |x[n]|^2 = \frac{1}{N_0} \sum_{n=m_0}^{m_0+N_0-1} |x[n]|^2 \qquad \text{for any integer } m_0$$

Example 4.5

find the energy of the signal $x[n] = (1/2)^n u[n]$

Solution:

$$E_x = \sum_{n=-\infty}^{\infty} \left|\left(\tfrac{1}{2}\right)^n u[n]\right|^2 = \sum_{n=0}^{\infty} \left|\left(\tfrac{1}{2}\right)^n\right|^2 = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{2n}$$

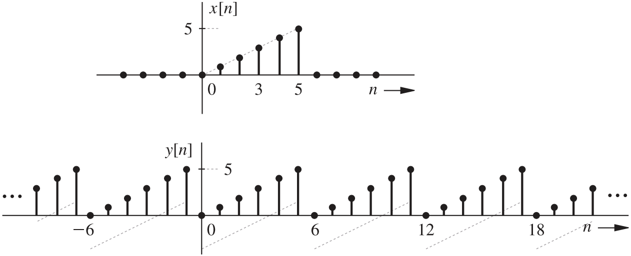
$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{4}\right)^n = 1 + \tfrac{1}{4} + \tfrac{1}{4^2} + \cdots$$

using the formula, $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, $|r| < 1$, we obtain

$$E_x = \frac{1}{1 - 1/4} = \frac{4}{3}$$

Example 4.6

find the energy of $x[n]$ and the power of the periodic signal $y[n]$ shown below



Solution:

$$E_x = \sum_{n=0}^{5} n^2 = 55$$

the period of signal y is $N_0 = 6$, hence

$$P_y = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |y[n]|^2 = \frac{1}{6} \sum_{n=0}^{5} n^2 = \frac{55}{6}$$

Example 4.7

find the energy E_x and power P_x of the signal $x[n] = 3\cos(\pi n/4)$

Solution: notice that $x[n]$ is 8-periodic and, therefore, a power signal:

$$P_x = \frac{1}{8}\sum_{n=0}^{7} |x[n]|^2 = \frac{1}{8}\left[2(3)^2 + 4(3/\sqrt{(2)})^2\right]$$

$$= \frac{1}{8}[18 + 18] = \frac{9}{2} = 4.5$$

since $0 < P_x < \infty$, we know that $E_x = \infty$

we can verify this power calculation in Matlab:

```
x = @(n) 3*cos(pi*n/4); n = 0:7;
Px = sum((x(n)).^2)/8
```

[output is `Px = 4.5000`]

Exercises

- show that the signal $x[n] = a^n u[n]$ is
  - an energy signal with energy $E_x = 1/(1 - |a|^2)$ if $|a| < 1$
  - a power signal with power $P_x = 1/2$ if $|a| = 1$
  - neither an energy signal nor a power signal if $|a| > 1$
- show that the power of a signal $De^{j(2\pi/N_0)n}$ is $|D|^2$

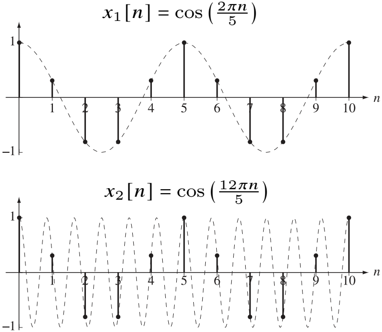
Outline

- DT signals
- signal operations
- useful DT signals
- signal energy and power
- **aliasing and DT sinusoids**

Nonuniqueness of DT sinusoids

- observe that $\cos(\Omega n) = \cos[(\Omega + 2\pi m)n]$ for integer m
- therefore, two DT sinusoids with frequencies Ω and $\Omega + 2\pi m$ are identical

Example:



$$x_2[n] = \cos(\frac{12\pi n}{5}) = \cos(\frac{2\pi n}{5} + 2\pi n) = x_1[n]$$

Fundamental band of DT sinusoids

- because $e^{j2\pi m} = 1$ for all integer values of m, we have $e^{j\Omega n} = e^{j(\Omega \pm 2\pi m)n}$ for any integer m
- the DT exponenetial $e^{j\Omega n}$ (or sinusoid) has a unique waveform only in a range separated by 2π

Fundamental band

- the values of Ω in the range $-\pi$ to π is called the *fundamental band*
- every frequency Ω, no matter how large, is identical to some frequency, Ω_a, in the fundamental band $(-\pi \leq \Omega_a < \pi)$, where

$$\Omega_a = \Omega - 2\pi m \quad -\pi \leq \Omega_a < \pi \quad \text{and} \quad m \text{ integer}$$

Apparent frequency

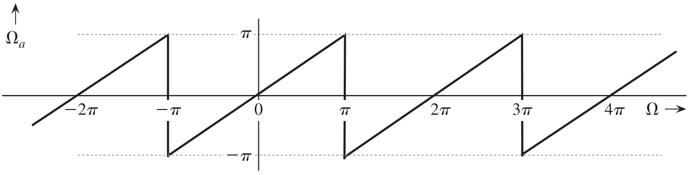
the *fundamental* or *apparent* frequency for a discrete-time sinusoid with frequency Ω is equal to $|\Omega_a|$ (the value in the range 0 to π that gives an equivalent siunsoid)

- example: $\cos(8.7\pi n + \theta) = \cos(0.7\pi n + \theta)$, so apparent frequency is $|\Omega_a| = 0.7\pi$

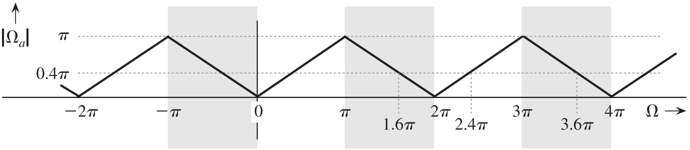
- since $\cos(-\Omega n + \theta) = \cos(\Omega n - \theta)$, a frequency in the range $-\pi$ to 0 is identical to the frequency (of the same magnitude) in the range 0 to π (but with a change in phase sign)

- example: $\cos(9.6\pi n + \theta) = \cos(-0.4\pi n + \theta) = \cos(0.4\pi n - \theta)$, so the apparent frequency is $|\Omega_a| = 0.4\pi$

the plot below shows the fundamental band frequency Ω_a versus the frequency Ω of a sinusoid; the frequency Ω_a is modulo 2π value of Ω



(a)



(b)

Example 4.8

express the following signals in terms of their apparent frequencies:

(a) $\cos(0.5\pi n + \theta)$

(b) $\cos(1.6\pi n + \theta)$

(c) $\sin(1.6\pi n + \theta)$

(d) $\cos(2.3\pi n + \theta)$

(e) $\cos(34.699n + \theta)$

Solution:

(a) $\Omega = 0.5\pi$ is in the reduced range already; because $\Omega_a = 0.5\pi$, there is no phase reversal, and the apparent sinusoid is $\cos(0.5\pi n + \theta)$

(b) $1.6\pi = -0.4\pi + 2\pi$ so that $\Omega_a = -0.4\pi$ and $|\Omega_a| = 0.4$; also, Ω_a is negative, implying sign change for the phase; hence, the apparent sinusoid is $\cos(0.4\pi n - \theta)$

(c) we first convert the sine to cosine $\sin(1.6\pi n + \theta) = \cos(1.6\pi n - (\pi/2) + \theta)$; in part (b), we found $\Omega_a = -0.4\pi$; hence, the apparent sinusoid is $\cos(0.4\pi n + (\pi/2) - \theta) = -\sin(0.4\pi n - \theta)$; in this case, both the phase and the amplitude change signs

(d) $2.3\pi = 0.3\pi + 2\pi$ so that $\Omega_a = 0.3\pi$; hence, the apparent sinusoid is $\cos(0.3\pi n + \theta)$

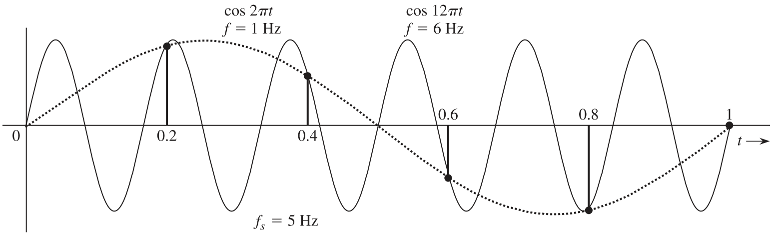
(e) we have $34.699 = -3 + 6(2\pi)$; hence, $\Omega_a = -3$, and the apparent frequency $|\Omega_a| = 3 \text{ rad/sample}$; because Ω_a is negative, there is a sign change of the phase and the apparent sinusoid is $\cos(3n - \theta)$

Aliasing and sampling rate

- a continuous-time sinusoid $\cos \omega t$ sampled every T seconds $(t = nT)$ results in a discrete-time sinusoid $\cos \omega nT$, which is $\cos \Omega n$ with $\Omega = \omega T$
- the discrete-time sinusoids $\cos \Omega n$ have unique waveforms only for the values of frequencies in the range $\Omega < \pi$ or $\omega T < \pi$
- therefore, samples of continuous-time sinusoids of two (or more) different frequencies can generate the same discrete-time signal
- this phenomenon is known as **aliasing** because through sampling, two entirely different analog sinusoids take on the same ”discrete-time” identity

Example: samples of two sinusoids $\cos 12\pi t$ and $\cos 2\pi t$ taken every 0.2 second

- the corresponding discrete-time frequencies $(\Omega = \omega T = 0.2\omega)$ are $\cos 2.4\pi$ and $\cos 0.4\pi$
- the apparent frequency of 2.4π is 0.4π, identical to the discrete-time frequency corresponding to the lower sinusoid



- aliasing causes ambiguity in digital signal processing, which makes it impossible to determine the true frequency of the sampled signal
- for instance, digitally processing a continuous-time signal that contains two distinct components of frequencies ω_1 and $\omega_2 > \omega_1$; if $\omega_2 - \omega_1 = 2k\pi/T)$, the the sampled frequencies $\Omega_1 = \omega_1 T$ and $\Omega_2 = \omega_2 T$ will be read as the same (lower of the two) frequency by the digital processor; as a result, the higher-frequency component ω_2 not only is lost for good, but also it reincarnates as a component of frequency ω_1, thus distorting the true amplitude of the original component of frequency ω_1

Avoiding aliasing

- to avoid aliasing, the frequencies of the continuous-time sinusoids to be processed should be kept within the fundamental band $\omega T \leq \pi$ or $\omega \leq \pi/T$
- this is because any continuous-time sinusoid of frequency in this range has a unique waveform when it is sampled

if $\omega_h = 2\pi f_h$ is the highest frequency to be processed, then, to avoid aliasing,

$$f_h < \frac{1}{2T} \quad \text{or} \quad T < \frac{1}{2f_h}$$

since the sampling frequency f_s is the reciprocal of the sampling interval T, we can also express

$$f_s = \frac{1}{T} > 2f_h \quad \text{or} \quad f_h < \frac{f_s}{2}$$

this result is a special case of the well-known sampling theorem; It states that for a discrete-time system to process a continuous-time sinusoid, the sampling rate must be greater than twice the frequency (in hertz) of the sinusoid

Example 4.9

(a) Determine the maximum sampling interval T that can be used in a discrete-time oscillator that generates a sinusoid of $50\ \mathrm{kHz}$

Solution: $T < 1/(2f_h) = 10\ \mu\mathrm{s}$; the sampling frequency is $f_s = 1/T > 100$ kHz

(b) a discrete-time amplifier uses a sampling interval $T = 25`\mu\mathrm{s}$; what is the highest frequency of a signal that can be processed with this amplifier without aliasing?

Solution: $f_h < 1/2T = 20\ \mathrm{kHz}$

Exercise

state with reasons whether the following sinusoids are periodic; if periodic, find the fundamental period N_0, and determine whether the fundamental frequency is equal to the sinusoid's frequency

(a) $\cos(3\pi n/7)$

(b) $\cos(10n/7)$

(c) $\cos(\sqrt{\pi}n)$

(d) $\sin(2.35\pi n + 1)$

References

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press, chapter 3 (3.1–3.3)
- M. J. Roberts, *Signals and Systems: Analysis Using Transform Methods and MATLAB*, McGraw Hill, chapter 3