10. Analysis using z-transform

- solution of linear difference equations
- transfer function and zero-state response
- frequency response
- aliasing and digital signal processing

Solving linear difference equations

- the z-transform converts difference equations into algebraic equations that are readily solved to find the solution in the z-domain
- taking the inverse z-transform of the z-domain solution yields the desired time-domain solution

Example:

$$y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n]$$

(initial conditions $y[-1] = 11/6$, $y[-2] = 37/36$, and input $x[n] = (2)^{-n}u[n]$)

- using left-shift property requires a knowledge of auxiliary conditions $y[0], y[1], \ldots, y[N-1]$, which are typically not given
- to directly utilize the knowledge of initial conditions, it is more convenient to express the difference equation in delay form and use the right-shift property

the delay-form difference equation is

$$y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2]$$

here, $y[n-m]$ (or $x[n-m]$) means $y[n-m]u[n]$ (or $x[n-m]u[n]$); we have

$$\begin{aligned}
y[n]u[n] &\Longleftrightarrow Y(z) \\
y[n-1]u[n] &\Longleftrightarrow \frac{1}{z}Y(z) + y[-1] = \frac{1}{z}Y(z) + \frac{11}{6} \\
y[n-2]u[n] &\Longleftrightarrow \frac{1}{z^2}Y(z) + \frac{1}{z}y[-1] + y[-2] = \frac{1}{z^2}Y(z) + \frac{11}{6z} + \frac{37}{36}
\end{aligned}$$

noting that for causal input $x[n]$, $x[-1] = x[-2] = \cdots = x[-n] = 0$, hence $x[n-m]u[n] \Longleftrightarrow \frac{1}{z^m}X(z)$, we thus have

$$\begin{aligned}
x[n] = (2)^{-n}u[n] = (0.5)^n u[n] &\Longleftrightarrow \frac{z}{z-0.5} \\
x[n-1]u[n] &\Longleftrightarrow \frac{1}{z}X(z) = \frac{1}{z}\frac{z}{z-0.5} = \frac{1}{z-0.5} \\
x[n-2]u[n] &\Longleftrightarrow \frac{1}{z^2}X(z) = \frac{1}{z^2}X(z) = \frac{1}{z(z-0.5)}
\end{aligned}$$

taking the z-transform of the difference equation:

$$Y(z) - 5\left[\frac{1}{z}Y(z) + \frac{11}{6}\right] + 6\left[\frac{1}{z^2}Y(z) + \frac{11}{6z} + \frac{37}{36}\right] = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y(z) - \left(3 - \frac{11}{z}\right) = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

rearranging gives,

$$\frac{Y(z)}{z} = \frac{3z^2 - 9.5z + 10.5}{(z-0.5)(z-2)(z-3)} = \frac{(26/15)}{z-0.5} - \frac{(7/3)}{z-2} + \frac{(18/5)}{z-3}$$

therefore,

$$Y(z) = \tfrac{26}{15}\left(\frac{z}{z-0.5}\right) - \tfrac{7}{3}\left(\frac{z}{z-2}\right) + \tfrac{18}{5}\left(\frac{z}{z-3}\right)$$

and

$$y[n] = \left[\tfrac{26}{15}(0.5)^n - \tfrac{7}{3}(2)^n + \tfrac{18}{5}(3)^n\right]u[n]$$

Zero-input and zero-state components

- we can separate the solution into zero-input and zero-state components
- to do so, we separate the response into terms arising from the input and terms arising from initial conditions (IC)

in the previous example, we have

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y(z) = \underbrace{\left(3 - \frac{11}{z}\right)}_{\text{IC terms}} + \underbrace{\frac{(3z+5)}{z(z-0.5)}}_{\text{input terms}}$$

multiplying both sides by z^2 yields

$$\left(z^2 - 5z + 6\right) Y(z) = \underbrace{z(3z - 11)}_{\text{IC terms}} + \underbrace{\frac{z(3z+5)}{z-0.5}}_{\text{input terms}}$$

hence,

$$Y(z) = \underbrace{\frac{z(3z-11)}{z^2-5z+6}}_{\text{zero-input response}} + \underbrace{\frac{z(3z+5)}{(z-0.5)\,(z^2-5z+6)}}_{\text{zero-state response}}$$

we expand both terms on the right-hand side into modified partial fractions:

$$Y(z) = \underbrace{\left[5\left(\frac{z}{z-2}\right) - 2\left(\frac{z}{z-3}\right)\right]}_{\text{zero-input response}} + \underbrace{\left[\frac{26}{15}\left(\frac{z}{z-0.5}\right) - \frac{22}{3}\left(\frac{z}{z-2}\right) + \frac{28}{5}\left(\frac{z}{z-3}\right)\right]}_{\text{zero-state response}}$$

thus

$$\begin{aligned} y[n] &= \underbrace{(5(2)^n - 2(3)^n)\,u[n]}_{\text{zero-input response}} + \underbrace{\left(\frac{26}{15}(0.5)^n - \frac{22}{3}(2)^n + \frac{28}{5}(3)^n\right)u[n]}_{\text{zero-state response}} \\ &= \left[-\frac{7}{3}(2)^n + \frac{18}{5}(3)^n + \frac{26}{15}(0.5)^n\right]u[n] \end{aligned}$$

Exercises

- solve the following equation if the initial conditions $y[-1] = 2$, $y[-2] = 0$, and the input $x[n] = u[n]$:

$$y[n+2] - \frac{5}{6}y[n+1] + \frac{1}{6}y[n] = 5x[n+1] - x[n]$$

separate the response into zero-input and zero-state responses
Answer:

$$\begin{aligned} y[n] &= \underbrace{\left(3\left(\frac{1}{2}\right)^n - \frac{4}{3}\left(\frac{1}{3}\right)^n\right)u[n]}_{\text{zero-input response}} + \underbrace{\left(12 - 18\left(\frac{1}{2}\right)^n + 6\left(\frac{1}{3}\right)^n\right)u[n]}_{\text{zero-state response}} \\ &= \left[12 - 15\left(\frac{1}{2}\right)^n + \frac{14}{3}\left(\frac{1}{3}\right)^n\right]u[n] \end{aligned}$$

- solve the following equation if the auxiliary conditions are $y[0] = 1$, $y[1] = 2$, and the input $x[n] = u[n]$:

$$y[n] + 3y[n-1] + 2y[n-2] = x[n-1] + 3x[n-2]$$

Answer: $y[n] = \left[\frac{2}{3} + 2(-1)^n - \frac{5}{3}(-2)^n\right]u[n]$

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- solution of linear difference equations
- **transfer function and zero-state response**
- frequency response
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The transfer function

the transfer function of an LTID system with impulse response $h[n]$ is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- $H(z)$ is z-transform of impulse response $h[n]$
- the LTID system response $y[n]$ to an everlasting exponential z^n is

$$y[n] = h[n] * z^n = \sum_{m=-\infty}^{\infty} h[m] z^{n-m} = H(z) z^n$$

  for fixed z, the output $y[n] = H(z)z^n$ has same form as input z^n; this input is called *eigenfunction*
- an alternate definition of the transfer function $H(z)$ of an LTID system is

$$H(z) = \left. \frac{\text{output signal}}{\text{input signal}} \right|_{\text{input=exponential } z^n}$$

Zero-state response

taking z-transform of $y[n] = x[n] * h[n]$, we have

$$Y(z) = X(z)H(z)$$

- we can find zero state response by taking the inverse z-transform:

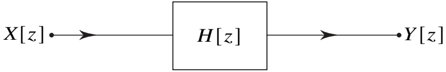
$$y[n] = \mathcal{Z}^{-1}\{X(z)H(z)\}$$

- given the input and output, we can find transfer function as

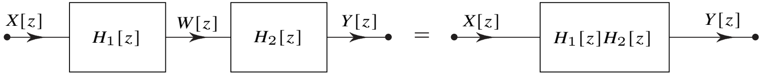
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\mathcal{Z}[\text{zero-state response}]}{\mathcal{Z}[\text{input}]}$$

Block diagrams

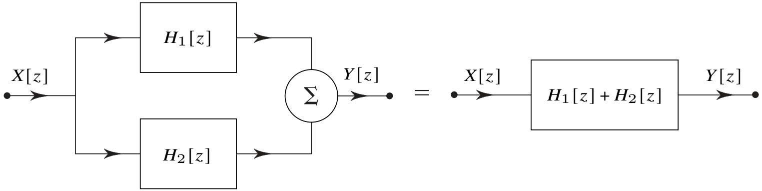
Block diagram of linear system



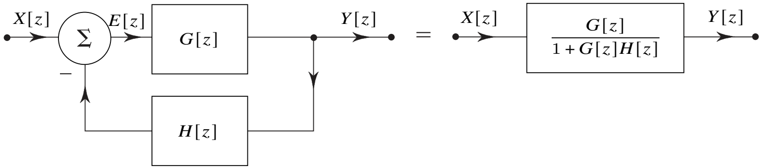
Cascade interconnection



Parallel interconnection



Feedback interconnection



$$\frac{Y(z)}{X(z)} = \frac{G(z)}{1 + G(z)H(z)}$$

Unit delay: the unit delay, which is represented by a box marked D, will be represented by its transfer function $1/z$

Transfer function of LTI difference system

Nth-order LTID system

$$Q[E]y[n] = P[E]x[n]$$

or

$$\left(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N\right) y[n]$$
$$= \left(b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1}E + b_N\right) x[n]$$

the transfer function is

$$H(z) = \frac{P(z)}{Q(z)} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1}z + b_N}{z^N + a_1 z^{N-1} + \cdots + a_{N-1}z + a_N}$$

Example 10.1

consider an LTID system described by the difference equation

$$y[n+2] + y[n+1] + 0.16y[n] = x[n+1] + 0.32x[n]$$

or

$$\left(E^2 + E + 0.16\right) y[n] = (E + 0.32)x[n]$$

find the transfer function and the zero-state response $y[n]$ due to input $x[n] = (-2)^{-n}u[n]$

Solution: from the difference equation, we find

$$H(z) = \frac{P(z)}{Q(z)} = \frac{z + 0.32}{z^2 + z + 0.16}$$

the input $x[n] = (-2)^{-n}u[n] = (-0.5)^n u[n]$ z-transform is

$$X(z) = \frac{z}{z + 0.5}$$

therefore,

$$Y(z) = X(z)H(z) = \frac{z(z+0.32)}{(z^2+z+0.16)\,(z+0.5)}$$

and

$$\begin{aligned}\frac{Y(z)}{z} &= \frac{(z+0.32)}{(z^2+z+0.16)\,(z+0.5)} = \frac{(z+0.32)}{(z+0.2)(z+0.8)(z+0.5)} \\ &= \frac{2/3}{z+0.2} - \frac{8/3}{z+0.8} + \frac{2}{z+0.5}\end{aligned}$$

so that

$$Y(z) = \frac{2}{3}\left(\frac{z}{z+0.2}\right) - \frac{8}{3}\left(\frac{z}{z+0.8}\right) + 2\left(\frac{z}{z+0.5}\right)$$

and

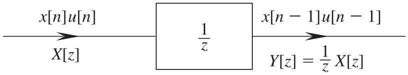
$$y[n] = \left[\frac{2}{3}(-0.2)^n - \frac{8}{3}(-0.8)^n + 2(-0.5)^n\right]u[n]$$

Example 10.2

if the input to the unit delay is $x[n]u[n]$, then its output is given by

$$y[n] = x[n-1]u[n-1]$$

show that the transfer function of a unit delay is $1/z$



Solution: the z-transform of this equation yields

$$Y(z) = \frac{1}{z}X(z) = H(z)X(z)$$

it follows that the transfer function of the unit delay is

$$H(z) = \frac{1}{z}$$

Stability

BIBO stability

- if all the poles of $H(z)$ are within the unit circle, then system is BIBO-stable (all the terms in $h[n]$ are decaying exponentials and $h[n]$ is absolutely summable)
- otherwise the system is BIBO-unstable

Internal stability: if $P(z)$ and $Q(z)$ do not share common factors, then the poles of $H(z)$ are the characteristic roots of the system; hence an LTID system is

1. asymptotically stable if and only if all the poles of its transfer function $H(z)$ are within the unit circle; the poles may be repeated or simple
2. unstable if and only if either one or both of the following conditions exist: (i) at least one pole of $H(z)$ is outside the unit circle; (ii) there are repeated poles of $H(z)$ on the unit circle
3. marginally stable if and only if there are no poles of $H(z)$ outside the unit circle, and there are some simple poles on the unit circle

Inverse systems

if $H(z)$ is the transfer function of a system $\mathcal{S}$, then $\mathcal{S}_i$, its inverse system, has a transfer function $H_i(z)$ given by

$$H_i(z) = \frac{1}{H(z)}$$

Examples:

- an accumulator $H(z) = z/(z-1)$ and a backward difference system $H_i(z) = (z-1)/z$ are inverse of each other
- if

  $$H(z) = \frac{z-0.4}{z-0.7}$$

  its inverse system transfer function is

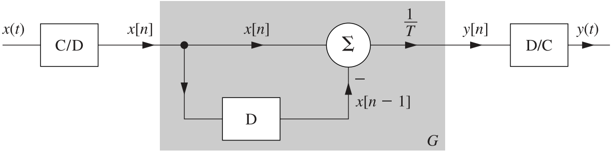
  $$H_i(z) = \frac{z-0.7}{z-0.4}$$

  as required by the property $H(z)H_i(z) = 1$; hence, it follows that

  $$h[n] * h_i[n] = \delta[n]$$

Exercises

- show that the transfer function of the digital differentiator (shaded block) is given by $H(z) = (z-1)/Tz$



- a discrete-time system is described by the following transfer function:

$$H(z) = \frac{z-0.5}{(z+0.5)(z-1)}$$

  (a) find the system response to input $x[n] = 3^{-(n+1)}u[n]$ and zero initial conditions
  (b) write the difference equation relating the output $y[n]$ to input $x[n]$ for this system

Answers:

(a) $y[n] = \frac{1}{3}\left[\frac{1}{2} - 0.8(-0.5)^n + 0.3\left(\frac{1}{3}\right)^n\right]u[n]$

(b) $y[n+2] - 0.5y[n+1] - 0.5y[n] = x[n+1] - 0.5x[n]$

Exercises

- find $h[n]$ by taking the inverse z-transform of $H(z)$ for the systems:
  (a) $y[n+1] - y[n] = x[n]$
  (b) $y[n] - 5y[n-1] + 6y[n-2] = 8x[n-1] - 19x[n-2]$
  (c) $y[n+2] - 4y[n+1] + 4y[n] = 2x[n+2] - 2x[n+1]$
  (d) $y[n] = 2x[n] - 2x[n-1]$
- show that an accumulator whose impulse response is $h[n] = u[n]$ is marginally stable but BIBO-unstable
- find the impulse responses of an accumulator and a first-order backward difference system; show that the convolution of the two impulse responses yields $\delta[n]$

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- solution of linear difference equations
- transfer function and zero-state response
- **frequency response**
- aliasing and digital signal processing

Frequency response

the LTID system response to complex sinusoid $x[n] = A_x e^{j\Omega n}$ is

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] A_x \, e^{j\Omega(n-m)} = H(e^{j\Omega}) A_x e^{j\Omega n}$$

$$= |H(e^{j\Omega})||A_x| e^{j\left(\Omega n + \angle H(e^{j\Omega}) + \angle A_x\right)}$$

- the system response to complex sinusoid is also complex sinusoid of the same frequency Ω multiplied by $H(e^{j\Omega})$
- $H(e^{j\Omega})$ is called the *frequency response* of the system, which is the transfer function $H(z) = \sum_{m=-\infty}^{\infty} h[m] z^{-m}$ evaluated at $z = e^{j\Omega}$
- using frequency response, we can find output for any sinusoidal input

Sinusoidal input response: the response to $\mathrm{Re}(e^{j(\Omega n + \theta)}) = \cos(\Omega n + \theta)$ is

$$y[n] = |H(e^{j\Omega})| \cos\left(\Omega n + \theta + \angle H(e^{j\Omega})\right)$$

Amplitude response

- $|H(e^{j\Omega})|$ is the amplitude gain of the system called *amplitude response* or *magnitude response*
- a plot of $|H(e^{j\Omega})|$ versus Ω shows the amplitude gain as a function of frequency Ω

Phase response

- $\angle H(e^{j\Omega})$ is the *phase response*
- a plot of $\angle H(e^{j\Omega})$ versus Ω shows how the system modifies or changes the phase of the input sinusoid

Steady-state response to causal inputs

- the response of an LTID system to a causal sinusoidal input $\cos(\Omega n)u[n]$ is $y[n]$, plus a natural component consisting of the characteristic modes
- for a stable system, the steady-state response of a system to a causal sinusoidal input $x[n] = \cos(\Omega n)u[n]$ is

$$y_{ss}[n] = \left|H(e^{j\Omega})\right| \cos\left(\Omega n + \angle H(e^{j\Omega})\right)$$

Response to sampled ct sinusoids

- in practice, the input may be a sampled continuous-time sinusoid $\cos \omega t$ (or an exponential $e^{j\omega t}$)
- when a sinusoid $\cos \omega t$ is sampled with sampling interval T, the resulting signal is a discrete-time sinusoid $\cos \omega nT$, obtained by setting $t = nT$ in $\cos \omega t$
- therefore, all the results developed here apply if we substitute ωT for Ω:

$$\Omega = \omega T$$

Example 10.3

for a system described by the equation

$$y[n+1] - 0.8y[n] = x[n+1]$$

find the system response to the inputs

(a) $x[n] = \cos(\frac{\pi}{6}n - 0.2)$

(b) $x[n] = 1$

(c) a sampled sinusoid $\cos(1500t)$ with sampling interval $T = 0.001$

Solution: the system equation can be expressed as

$$(E - 0.8)y[n] = Ex[n]$$

therefore, the transfer function of the system is

$$H(z) = \frac{z}{z - 0.8} = \frac{1}{1 - 0.8z^{-1}}$$

the frequency response is

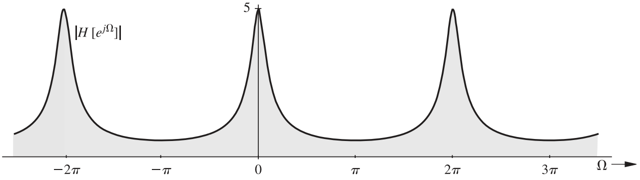
$$H(e^{j\Omega}) = \frac{1}{1 - 0.8e^{-j\Omega}} = \frac{1}{(1 - 0.8\cos\Omega) + j0.8\sin\Omega}$$

therefore,

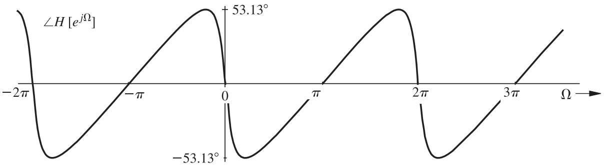
$$\left|H(e^{j\Omega})\right| = \frac{1}{\sqrt{(1 - 0.8\cos\Omega)^2 + (0.8\sin\Omega)^2}} = \frac{1}{\sqrt{1.64 - 1.6\cos\Omega}}$$

and

$$\angle H(e^{j\Omega}) = -\tan^{-1}\left[\frac{0.8\sin\Omega}{1 - 0.8\cos\Omega}\right]$$


5
$|H[e^{j\Omega}]|$
-2π
$-\pi$
0
π
2π
3π
Ω

(a)


53.13°
$\angle H[e^{j\Omega}]$
-2π
$-\pi$
0
π
2π
3π
Ω
−53.13°

(b)

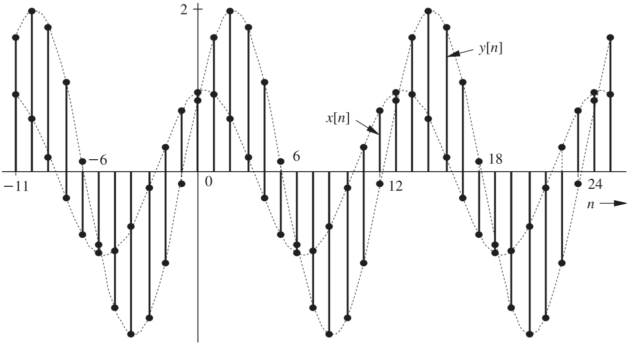
(a) for $x[n] = \cos[(\pi/6)n - 0.2], \Omega = \pi/6$ and

$$\left|H\left(e^{j\pi/6}\right)\right| = \frac{1}{\sqrt{1.64 - 1.6\cos\frac{\pi}{6}}} = 1.983$$

$$\angle H\left(e^{j\pi/6}\right) = -\tan^{-1}\left[\frac{0.8\sin\frac{\pi}{6}}{1 - 0.8\cos\frac{\pi}{6}}\right] = -0.916 \text{ rad}$$

therefore,

$$y[n] = 1.983\cos\left(\frac{\pi}{6}n - 0.2 - 0.916\right) = 1.983\cos\left(\frac{\pi}{6}n - 1.116\right)$$



(b) since $1^n = (e^{j\Omega})^n$ with $\Omega = 0$, the amplitude response is

$$H\left(e^{j0}\right) = \frac{1}{\sqrt{1.64 - 1.6\cos(0)}} = \frac{1}{\sqrt{0.04}} = 5 = 5\angle 0$$

therefore,

$$\left|H\left(e^{j0}\right)\right| = 5 \quad \text{and} \quad \angle H\left(e^{j0}\right) = 0$$

and the system response to input 1 is

$$y[n] = 5\left(1^n\right) = 5 \quad \text{for all } n$$

(c) a sinusoid $\cos 1500t$ sampled every T seconds $(t = nT)$ results in a discrete-time sinusoid

$$x[n] = \cos 1500nT$$

for $T = 0.001$, the input is

$$x[n] = \cos(1.5n)$$

in this case, $\Omega = 1.5$ and

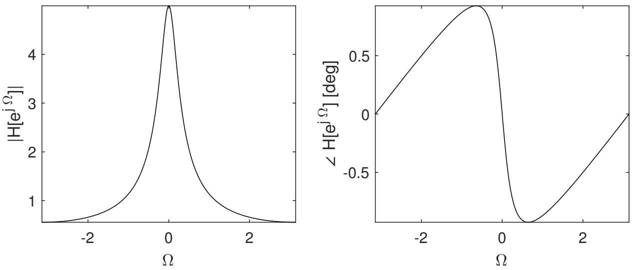
$$\left|H\left(e^{j1.5}\right)\right| = \frac{1}{\sqrt{1.64 - 1.6\cos(1.5)}} = 0.809$$
$$\angle H\left(e^{j1.5}\right) = -\tan^{-1}\left[\frac{0.8\sin(1.5)}{1 - 0.8\cos(1.5)}\right] = -0.702\text{rad}$$

therefore,

$$y[n] = 0.809\cos(1.5n - 0.702)$$

Frequency response using MATLAB

```
Omega = linspace(-pi,pi,400); H = @(z) z./(z-0.8);
subplot(1,2,1); plot(Omega,abs(H(exp(1j*Omega))),'k'); axis tight;
xlabel('\Omega'); ylabel('|H(e^{j \Omega})|');
subplot(1,2,2); plot(Omega,angle(H(exp(1j*Omega))*180/pi),'k');
axis tight;
xlabel('\Omega'); ylabel('\angle H(e^{j \Omega}) [deg]');
```



Exercises

- for a system specified by the equation

$$y[n+1] - 0.5y[n] = x[n]$$

find the amplitude and the phase response; find the system response to sinusoidal input $\cos[1000t - (\pi/3)]$ sampled every $T = 0.5\ \mathrm{ms}$

Answer:

$$\left|H(e^{j\Omega})\right| = \frac{1}{\sqrt{1.25 - \cos\Omega}}$$

$$\angle H(e^{j\Omega}) = -\tan^{-1}\left[\frac{\sin\Omega}{\cos\Omega - 0.5}\right]$$

$$y[n] = 1.639\cos\left(0.5n - \frac{\pi}{3} - 0.904\right) = 1.639\cos(0.5n - 1.951)$$

- show that for an ideal delay $(H(z) = 1/z)$, the amplitude response $\left|H(e^{j\Omega})\right| = 1$, and the phase response $\angle H(e^{j\Omega}) = -\Omega$; thus, for an ideal delay, the phase shift of the output sinusoid is proportional to the frequency of the input sinusoid (linear phase shift)

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Periodic nature of frequency response

the frequency response $H(e^{j\Omega})$ is a periodic function of Ω with period 2π

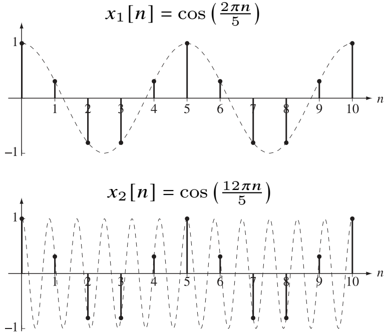
$$H(e^{j\Omega}) = H(e^{j(\Omega+2\pi m)}) \quad m \text{ integer}$$

- because $e^{j2\pi m} = 1$ for all integer values of m, we have $e^{j\Omega n} = e^{j(\Omega \pm 2\pi m)n}$ for any integer m:
- the DT exponenetial $e^{j\Omega n}$ (or sinusoid) has a unique waveform only in a range separated by 2π

Nonuniqueness of DT sinusoids

- observe that $\cos(\Omega n) = \cos[(\Omega + 2\pi m)n]$ for integer m $[e^{j\Omega n} = e^{j(\Omega \pm 2\pi m)n}]$
- any two DT sinusoids with frequencies Ω and $\Omega + 2\pi m$ are identical

Example: $x_1[n] = \cos(\frac{2\pi n}{5})$, $x_2[n] = \cos(\frac{12\pi n}{5}) = \cos(\frac{2\pi n}{5} + 2\pi n) = x_1[n]$



Fundamental (apparent) frequency

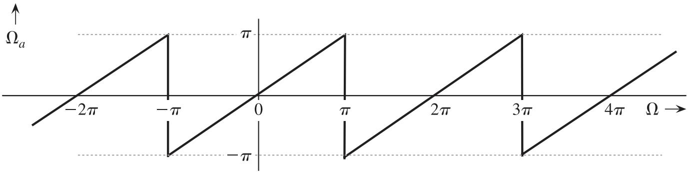
- DT sinusoid has a unique waveform only in a range separated by 2π
- the values of Ω in the range $-\pi$ to π is called the *fundamental band*
- every frequency Ω is identical to some frequency, Ω_a:

$$\Omega_a = \Omega - 2\pi m \quad -\pi \leq \Omega_a < \pi \quad \text{and} \quad m \text{ integer}$$

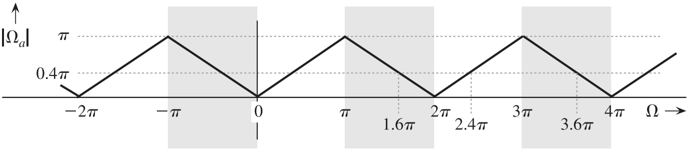
Apparent frequency the *fundamental* or *apparent* frequency for a DT sinusoid is equal to $|\Omega_a|$

- example: $\cos(8.7\pi n + \theta) = \cos(0.7\pi n + \theta)$, so $|\Omega_a| = 0.7\pi$
- since $\cos(-\Omega n + \theta) = \cos(\Omega n - \theta)$, a frequency in the range $-\pi$ to 0 is identical to the frequency in the range 0 to π (with a change in phase sign)
- example: $\cos(9.6\pi n + \theta) = \cos(-0.4\pi n + \theta) = \cos(0.4\pi n - \theta)$, so $|\Omega_a| = 0.4\pi$

the plot below shows the fundamental band frequency Ω_a versus the frequency Ω of a sinusoid; the frequency Ω_a is modulo 2π value of Ω



(a)



(b)

Example 10.4

express the following signals in terms of their apparent frequencies:

(a) $\cos(0.5\pi n + \theta)$

(b) $\cos(1.6\pi n + \theta)$

(c) $\sin(1.6\pi n + \theta)$

(d) $\cos(2.3\pi n + \theta)$

(e) $\cos(34.699 n + \theta)$

Solution:

(a) $\Omega = 0.5\pi$ is in the reduced range already; because $\Omega_a = 0.5\pi$, there is no phase reversal, and the apparent sinusoid is $\cos(0.5\pi n + \theta)$

(b) $1.6\pi = -0.4\pi + 2\pi$ so that $\Omega_a = -0.4\pi$ and $|\Omega_a| = 0.4$; also, Ω_a is negative, implying sign change for the phase; hence, the apparent sinusoid is $\cos(0.4\pi n - \theta)$

(c) we first convert the sine to cosine $\sin(1.6\pi n + \theta) = \cos(1.6\pi n - (\pi/2) + \theta)$; in part (b), we found $\Omega_a = -0.4\pi$; hence, the apparent sinusoid is $\cos(0.4\pi n + (\pi/2) - \theta) = -\sin(0.4\pi n - \theta)$; in this case, both the phase and the amplitude change signs

(d) $2.3\pi = 0.3\pi + 2\pi$ so that $\Omega_a = 0.3\pi$; hence, the apparent sinusoid is $\cos(0.3\pi n + \theta)$

(e) we have $34.699 = -3 + 6(2\pi)$; hence, $\Omega_a = -3$, and the apparent frequency $|\Omega_a| = 3 \text{ rad/sample}$; because Ω_a is negative, there is a sign change of the phase and the apparent sinusoid is $\cos(3n - \theta)$

Aliasing and sampling rate

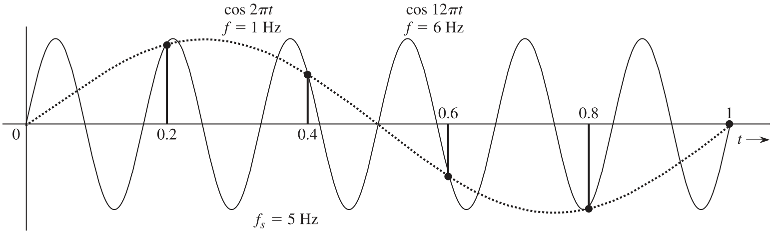
a CT sinusoid $\cos\omega t$ sampled every T seconds $(t = nT)$ results in a DT sinusoid

$$\cos\Omega n \quad \text{where} \quad \Omega = \omega T$$

- since DT sinusoids $\cos\Omega n$ have unique waveforms only for the values of frequencies in the range $\Omega < \pi$ ($\omega T < \pi$), samples of continuous-time sinusoids of two (or more) different frequencies can generate the same discrete-time signal
- this phenomenon is known as **aliasing** because through sampling, two entirely different analog sinusoids take on the same "discrete-time" identity

Example: two sinusoids $\cos 12\pi t$ and $\cos 2\pi t$ sampled every 0.2 second

- the sampled DT sinusoids $(\Omega = \omega T = 0.2\omega)$ are $\cos 2.4\pi n$ and $\cos 0.4\pi n$
- the apparent frequency of 2.4π is 0.4π, identical to the discrete-time frequency corresponding to the lower sinusoid



aliasing causes ambiguity in digital signal processing, which makes it impossible to determine the true frequency of the sampled signal

Avoiding aliasing (sampling theorem)

to avoid aliasing, the frequencies of the continuous-time sinusoids to be sampled must be kept within the fundamental band $\omega T \leq \pi$

Sampling theorem: if $\omega_h = 2\pi f_h$ is the highest frequency to be processed, then, to avoid aliasing,

$$T < \frac{1}{2f_h}$$

since the sampling freq. f_s is the reciprocal of the sampling interval T, we have

$$f_s = \frac{1}{T} > 2f_h$$

(this result is a special case of the well-known sampling theorem)

Example 10.5

(a) determine the maximum sampling interval T that can be used in a discrete-time oscillator that generates a sinusoid of $50\ \mathrm{kHz}$

Solution: $T < 1/(2f_h) = 10\ \mu\mathrm{s}$; the sampling frequency is $f_s = 1/T > 100$ kHz

(b) a discrete-time amplifier uses a sampling interval $T = 25\ \mu\mathrm{s}$; what is the highest frequency of a signal that can be processed with this amplifier without aliasing?

Solution: $f_h < 1/2T = 20\ \mathrm{kHz}$

References

Reference:

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press, chapter 5.3–5.5.

Further reading and practice exercises:

- Read section(s) 5.3–5.5 in the book.