6. Time-domain analysis of discrete-time systems

- zero-input response
- unit-impulse response
- zero-state response and convolution
- system stability

Difference equation

Advance-form

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_M x[n]$$

- time-invariant if coefficients a_i, b_i are constants (independent of n)
- causal if $M \leq N$

Causal delay-form: let $M = N$ and replace all n by $n - N$:

$$y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{N-1} x[n-N+1] + b_N x[n-N]$$

- delay form is more natural form since delay operation is realizable
- advance form is more mathematically convenience compared to delay form

LTI difference system

Operator notation: for discrete-time systems, the notation E is used to denote the operation for advancing a sequence by one time unit

- $Ex[n] \triangleq x[n+1]$
- $E^k x[n] \triangleq x[n+k]$

LTID (difference) system: the advance-form difference equation with $M = N$ can be expressed as

$$Q[E]y[n] = P[E]x[n] \tag{6.1}$$

where $Q[E]$ and $P[E]$ are Nth-order polynomial operators

$$Q[E] = E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N$$
$$P[E] = b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1}E + b_N$$

Zero-input response

the *zero-input response* $y_0[n]$ is the solution of (6.1) with $x[n] = 0$:

$$\underbrace{\left(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N\right)}_{Q[E]} y_0[n] = 0 \tag{6.2}$$

- a linear combination of $y_0[n]$ and advanced $y_0[n]$ is zero for all n
- possible if and only if $y_0[n]$ and advanced $y_0[n]$ share the same form; only an exponential function γ^n has this property ($E^k\{\gamma^n\} = \gamma^k\gamma^n$)
- let $y_0[n] = c\gamma^n$, then using $E^k y_0[n] = c\gamma^{n+k}$ in (6.2), we obtain

$$c\left(\gamma^N + a_1\gamma^{N-1} + \cdots + a_{N-1}\gamma + a_N\right)\gamma^n = cQ[\gamma] = 0$$

hence, $c\gamma^n$ satisfies the zero-input difference equatio (6.2) if $Q[\gamma] = 0$

Characteristic equation

$$Q[\gamma] = \gamma^N + a_1\gamma^{N-1} + \cdots + a_{N-1}\gamma + a_N = 0$$

- $Q[\gamma]$ is the *characteristic polynomial*
- $Q[\gamma] = 0$ has N solutions $\gamma_1, \gamma_2, \ldots, \gamma_N$ called *characteristic roots* of the system or *characteristic values* (also *eigenvalues*) of the system
- all $c_1\gamma_1^n, c_2\gamma_2^n, \ldots, c_N\gamma_N^n$ satisfy the zero-input difference equation
- the general form of the zero-input response depends on whether the roots are distinct or repeated

Zero-input response

Distinct roots: for distinct roots, $\gamma_1, \ldots, \gamma_N$, the zero input solution is

$$y_0[n] = c_1\gamma_1^n + c_2\gamma_2^n + \cdots + c_N\gamma_N^n$$

- $\gamma_1, \ldots, \gamma_N$ are the *characteristic modes* or *natural modes* of the system
- $c_1, c_2, \ldots, c_N$ are constants determined from N auxiliary conditions (*e.g.*, initial conditions)

Repeated roots: if the characteristic polynomial has a repeated root:

$$Q[\gamma] = (\gamma - \gamma_1)^r\,(\gamma - \gamma_{r+1})\,(\gamma - \gamma_{r+2})\cdots(\gamma - \gamma_N)$$

then the zero-input response of the system is

$$y_0[n] = \left(c_1 + c_2 n + \cdots + c_r n^{r-1}\right)\gamma_1^n + \sum_{i=r+1}^{N} c_i\gamma_i^n$$

- root γ_1 repeats r times (root of multiplicity r)
- the characteristic modes for γ_1 are $\gamma_1^n, n\gamma_1^n, n^2\gamma_1^n, \ldots, n^{r-1}\gamma_1^n$

Example 6.1

determine the zero-input response $y_0[n]$ of

$$y[n+2] - 0.6y[n+1] - 0.16y[n] = 5x[n+2]$$

with input $x[n] = 4^{-n}u[n]$ and initial conditions $y[-1] = 0$ and $y[-2] = 25/4$

Solution: the system of equation in operator notation is

$$\left(E^2 - 0.6E - 0.16\right) y[n] = 5E^2 x[n]$$

the characteristic polynomial is

$$Q[\gamma] = \gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma - 0.8)$$

the characteristic equation is

$$(\gamma + 0.2)(\gamma - 0.8) = 0$$

the characteristic roots are $\gamma_1 = -0.2$ and $\gamma_2 = 0.8$

the zero-input response is

$$y_0[n] = c_1(-0.2)^n + c_2(0.8)^n$$

to find the constants c_1 and c_2, we set $n = -1$ and -2 in the previous equation and use $y_0[-1] = 0$ and $y_0[-2] = 25/4$ to obtain:

$$\begin{aligned} 0 &= -5c_1 + \frac{5}{4}c_2 \\ \tfrac{25}{4} &= 25c_1 + \tfrac{25}{16}c_2 \end{aligned}$$

solving gives $c_1 = \frac{1}{5}$ and $c_2 = \frac{4}{5}$; therefore

$$y_0[n] = \tfrac{1}{5}(-0.2)^n + \tfrac{4}{5}(0.8)^n, \quad n \geq 0$$

Example 6.2

$$\left(E^2 + 6E + 9\right) y[n] = \left(2E^2 + 6E\right) x[n]$$

determine the zero-input response $y_0[n]$ if $y_0[-1] = -1/3$ and $y_0[-2] = -2/9$

Solution: the characteristic polynomial is $\gamma^2 + 6\gamma + 9 = (\gamma + 3)^2$, and we have a repeated characteristic root at $\gamma = -3$; the characteristic modes are $(-3)^n$ and $n(-3)^n$; hence, the zero-input response is

$$y_0[n] = (c_1 + c_2 n)\,(-3)^n$$

we can determine the constants c_1 and c_2 from the initial conditions, doing so we get $c_1 = 4$ and $c_2 = 3$; hence

$$y_0[n] = (4 + 3n)(-3)^n \quad n \geq 0$$

Complex roots

for difference equation with real coefficients, complex roots appear as conjugates pairs:

$$\gamma = |\gamma| e^{j\beta} \quad \text{and} \quad \gamma^* = |\gamma| e^{-j\beta}$$

complex form: the zero-input response is

$$y_0[n] = c_1 \gamma^n + c_2 (\gamma^*)^n = c_1 |\gamma|^n e^{j\beta n} + c_2 |\gamma|^n e^{-j\beta n}$$

real-form: let $c_1 = \frac{c}{2} e^{j\theta}$ and $c_2 = \frac{c}{2} e^{-j\theta}$, then we can write output as

$$y_0[n] = c|\gamma|^n \cos(\beta n + \theta)$$

where c and θ are constants determined from the auxiliary conditions

Example 6.3

$$\left(E^2 - 1.56E + 0.81\right) y[n] = (E + 3)x[n]$$

determine the zero-input response $y_0[n]$ if $y_0[-1] = 2$ and $y_0[-2] = 1$

Solution: the characteristic equation is $\left(\gamma^2 - 1.56\gamma + 0.81\right) = 0$ and the characteristic roots are $0.78 \pm j0.45 = 0.9e^{\pm j(\pi/6)}$; so the complex form solution:

$$y_0[n] = c(0.9)^n e^{j\pi n/6} + c^*(0.9)^n e^{-j\pi n/6}$$

using the initial conditions $y_0[-1] = 2$ and $y_0[-2] = 1$, we find

$$c = 1.1550 - j0.2025 = 1.1726e^{-j0.1735}$$

$$c^* = 1.1550 + j0.2025 = 1.1726e^{j0.1735}$$

hence

$$y_0[n] = 1.1726e^{-j0.1735}(0.9)^n e^{j\pi n/6} + 1.1726e^{j0.1735}(0.9)^n e^{-j\pi n/6}$$

we can also find $y_0[n]$ using the real form of the solution; since $\gamma = 0.9e^{\pm j(\pi/6)}$, we have $|\gamma| = 0.9$ and $\beta = \pi/6$, and the real-form zero-input response is

$$y_0[n] = c(0.9)^n \cos(\tfrac{\pi}{6}n + \theta)$$

to determine the constants c and θ, we use the initial conditions:

$$y_0[-1] = \frac{c}{0.9}\cos(-\tfrac{\pi}{6} + \theta) = \frac{\sqrt{3}}{1.8}c\cos\theta + \frac{1}{1.8}c\sin\theta = 2$$

$$y_0[-2] = \frac{c}{(0.9)^2}\cos(-\tfrac{\pi}{3} + \theta) = \frac{1}{1.62}c\cos\theta + \frac{\sqrt{3}}{1.62}c\sin\theta = 1$$

solving gives $c\cos\theta = 2.308$ and $c\sin\theta = -0.397$; hence

$$\tan\theta = \frac{c\sin\theta}{c\cos\theta} = \frac{-0.397}{2.308} = -0.172, \qquad \theta = \tan^{-1}(-0.172) = -0.17\ \mathrm{rad}$$

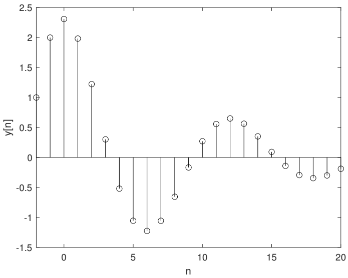
substituting $\theta = -0.17$ radian in $c\cos\theta = 2.308$ yields $c = 2.34$ and

$$y_0[n] = 2.34(0.9)^n \cos(\tfrac{\pi}{6}n - 0.17) \quad n \geq 0$$

Finding zero-input response iteratively using MATLAB

use MATLAB to iteratively compute and then plot the zero-input response for $\left(E^2 - 1.56E + 0.81\right) y[n] = (E+3)x[n]$ with $y[-1] = 2$ and $y[-2] = 1$

```
n = (-2:20)'; y = [1;2;zeros(length(n)-2,1)];
for k = 1:length(n)-2,
  y(k+2) = 1.56*y(k+1)-0.81*y(k);
end;
clf; stem(n,y,'k'); xlabel('n'); ylabel('y[n]');
axis([-2 20 -1.5 2.5]);
```



Exercises

find and sketch the zero-input response for the systems described by the following equations:

(a) $y[n+1] - 0.8y[n] = 3x[n+1]$ with initial condition $y[-1] = 10$

(b) $y[n+1] + 0.8y[n] = 3x[n+1]$ with initial condition $y[-1] = 10$

(c) $y[n] + 0.3y[n-1] - 0.1y[n-2] = x[n] + 2x[n-1]$ with initial conditions $y_0[-1] = 1$ and $y_0[-2] = 33$

(d) $y[n] + 4y[n-2] = 2x[n]$ with $y[-1] = -1/(2\sqrt{2})$ and $y[-2] = 1/(4\sqrt{2})$

in each case verify the solutions by computing the first three terms using the iterative method

Answers:

(a) $8(0.8)^n$

(b) $-8(-0.8)^n$

(c) $y_0[n] = (0.2)^n + 2(-0.5)^n$

(d) $y_0[n] = (2)^n \cos\left(\frac{\pi}{2}n - \frac{3\pi}{4}\right)$

Outline

- zero-input response
- **unit-impulse response**
- zero-state response and convolution
- system stability

Impulse response

- the (unit) *impulse response* $h[n]$ is output of the system when the input is $\delta[n]$ with zero zero initial conditions
- an LTI system is causal if and only if $h[n] = 0$ for $n < 0$

Linear difference system

$$\begin{aligned} &\underbrace{\left(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N\right)}_{Q[E]} y[n] \\ &= \underbrace{\left(b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1}E + b_N\right)}_{P[E]} x[n] \end{aligned} \tag{6.3}$$

the impulse response $h[n]$ to the above difference system satisfies:

- $Q[E]h[n] = P[E]\delta[n]$
- subject to initial conditions

$$h[-1] = h[-2] = \cdots = h[-N] = 0$$

Example 6.4

iteratively compute the first two values of the impulse response $h[n]$ of:

$$y[n] - 0.6y[n-1] - 0.16y[n-2] = 5x[n]$$

Solution: letting the input $x[n] = \delta[n]$ and the output $y[n] = h[n]$, we have

$$h[n] = 0.6h[n-1] + 0.16h[n-2] + 5\delta[n]$$

let $h[-1] = h[-2] = 0$; setting $n = 0$ in this equation yields

$$h[0] = 0.6(0) + 0.16(0) + 5(1) = 5$$

setting $n = 1$ in the same equation and using $h[0] = 5$, we obtain

$$h[1] = 0.6(5) + 0.16(0) + 5(0) = 3$$

continuing this way, we can determine any number of terms of $h[n]$

Closed form expression

the impulse response to system (6.3) with $a_N \neq 0$ can be expressed as

$$h[n] = A_0\delta[n] + y_c[n]u[n]$$

- $A_0 = b_N/a_N$ (assuming $a_N \neq 0$)
- $y_c[n]$ is a linear combination of the characteristic modes
  - for unrepeated roots $y_c[n] = c_1\gamma_1^n + \cdots + c_N\gamma_N^n$
  - repeated roots has form as in page 6.6
- to find the N unknowns $c_1, \ldots, c_N$, we need to compute N values $h[0], h[1], h[2], \ldots, h[N-1]$ iteratively

Finding A_0: substituting the above into (6.3), we obtain

$$Q[E]\left(A_0\delta[n] + y_c[n]u[n]\right) = P[E]\delta[n]$$

since $y_c[n]$ is made up of characteristic modes, $Q[E]y_c[n] = 0$; hence

$$A_0\left(\delta[n+N] + a_1\delta[n+N-1] + \cdots + a_N\delta[n]\right) = b_0\delta[n+N] + \cdots + b_N\delta[n]$$

setting $n = 0$ and using $\delta[m] = 0$ for all $m \neq 0$, and $\delta[0] = 1$, we obtain

$$A_0 a_N = b_N \quad \Longrightarrow \quad A_0 = \frac{b_N}{a_N} \quad \text{(assuming } a_N \neq 0\text{)}$$

Example 6.5

determine the unit impulse response $h[n]$ for a system specified by the equation

$$y[n] - 0.6y[n-1] - 0.16y[n-2] = 5x[n]$$

Solution: this equation can be expressed in the advance form as

$$y[n+2] - 0.6y[n+1] - 0.16y[n] = 5x[n+2]$$

or in advance operator form as

$$\left(E^2 - 0.6E - 0.16\right) y[n] = 5E^2 x[n]$$

the characteristic polynomial is

$$\gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma - 0.8)$$

the characteristic modes are $(-0.2)^n$ and $(0.8)^n$; therefore,

$$y_c[n] = c_1(-0.2)^n + c_2(0.8)^n$$

by inspection, we see that $a_N = -0.16$ and $b_N = 0$; hence

$$h[n] = [c_1(-0.2)^n + c_2(0.8)^n]\, u[n]$$

to determine c_1 and c_2, we need to find two values of $h[n]$ iteratively; from the example in page 6.16, we know that $h[0] = 5$ and $h[1] = 3$; hence

$$\left.\begin{array}{l} h[0] = 5 = c_1 + c_2 \\ h[1] = 3 = -0.2c_1 + 0.8c_2 \end{array}\right\} \quad \Longrightarrow \quad \begin{array}{l} c_1 = 1 \\ c_2 = 4 \end{array}$$

therefore,

$$h[n] = [(-0.2)^n + 4(0.8)^n]\, u[n]$$

Other cases

when $a_N = 0$ and $a_{N-1} \neq 0$, then

$$h[n] = A_0\delta[n] + A_1\delta[n-1] + y_c[n]u[n]$$

- $y_c[n]$ contains the characteristic terms of $\hat{Q}[\gamma] = Q[\gamma]/\gamma$
- to find the unknowns $A_0, A_1, c_1, \ldots, c_N$, we need to compute $N+1$ values $h[0], h[1], h[2], \ldots, h[N]$ iteratively

when $a_N = a_{N-1} = 0$ and $a_{N-2} \neq 0$, then

$$h[n] = A_0\delta[n] + A_1\delta[n-1] + A_2\delta[n-2] + y_c[n]u[n]$$

- $y_c[n]$ contains the characteristic terms of $\hat{Q}[\gamma] = Q[\gamma]/\gamma^2$
- to find the unknowns $A_0, A_1, A_2, c_1, \ldots, c_N$, we need to compute $N+1$ values $h[0], h[1], h[2], \ldots, h[N]$ iteratively

...etc

Example 6.6

determine the impulse response $h[n]$ of a system described by the equation

$$(E^3 + E^2)y[n] = x[n]$$

Solution: in this case, $a_N = a_{N-1} = 0$, and the characteristic roots: one at -1 and two at 0; only the nonzero characteristic root shows up in $y_c[n]$, so

$$h[n] = A_0\delta[n] + A_1\delta[n-1] + A_2\delta[n-2] + c_1(-1)^n u[n]$$

to determine the coefficients A_0, A_1, A_2, and c_1, we require $N+1=4$ values of $h[n] (n \geq 0)$, which we obtain iteratively using Matlab

```
n = (-3:3); delta = (n==0); h = zeros(size(n));
for ind = find(n>=0),
   h(ind) = -h(ind-1)+delta(ind-3);
end
h(n>=0)
```

[output: `ans = 0 0 0 1`] using these values to solve for th constants, we get

$$h[n] = \delta[n] - \delta[n-1] + \delta[n-2] - (-1)^n u[n]$$

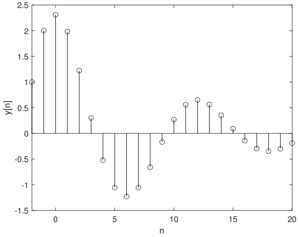
Solving LTI difference system in MATLAB

we can use the `filter` command in MATLAB to solve constant coefficient difference equations:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k]$$

Example: $y[n] - 0.6y[n-1] - 0.16y[n-2] = 5x[n]$ with $x[n] = \delta[n]$

```
n = (0:19); delta = @(n) 1.0.*(n==0);
a = [1 -0.6 -0.16]; b = [5 0 0];
h = filter(b,a,delta(n));
clf; stem(n,h,'k'); xlabel('n'); ylabel('h[n]');
```



Exercises

- find the unit impulse response $h[n]$ of the LTID systems
  (a) $y[n+1] - y[n] = x[n]$
  (b) $y[n] - 5y[n-1] + 6y[n-2] = 8x[n-1] - 19x[n-2]$
  (c) $y[n+2] - 4y[n+1] + 4y[n] = 2x[n+2] - 2x[n+1]$

  Answers:
  (a) $h[n] = u[n-1]$
  (b) $h[n] = -\frac{19}{6}\delta[n] + \left[\frac{3}{2}(2)^n + \frac{5}{3}(3)^n\right] u[n]$
  (c) $h[n] = (2+n)2^n u[n]$

- *Nonrecursive impulse response.* the find the iupulse response of
  $y[n] = 2x[n] - 2x[n-1]$
  Answer: $h[n] = 2\delta[n] - 2\delta[n-1]$

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Derivation of zero-state response

we can express any arbitrary input $x[n]$ as a sum of impulse components:

$$\begin{aligned} x[n] &= x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots \\ &\quad + x[-1]\delta[n+1] + x[-2]\delta[n+2] + \cdots \\ &= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \end{aligned}$$

let $h[n]$ be the system response to impulse input $\delta[n]$ ($\delta[n] \Longrightarrow h[n]$), then due to linearity and time invariance

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \Longrightarrow \underbrace{\sum_{m=-\infty}^{\infty} x[m]h[n-m]}_{y[n]}$$

the right-hand side is the system response $y[n]$ to input $x[n]$

Zero-state response and convolution

the zero-state response is:

$$y[n] = x[n] * h[n-m] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

- the summation is known as the *convolution sum of* $x[n]$ *and* $h[n]$
- for causal input and system ($h[k] = x[k] = 0$ for $k < 0$), we have

$$y[n] = \sum_{m=0}^{n} x[m]h[n-m]$$

- we use $*$ to denote the convolution sum two signals $x_1[n]$ and $x_2[n]$:

Example 6.7

determine $c[n] = x[n] * g[n]$ analytically for

$$x[n] = (0.8)^n u[n] \quad \text{and} \quad g[n] = (0.3)^n u[n]$$

Solution: note that $x[m] = (0.8)^m u[m]$ and $g[n-m] = (0.3)^{n-m} u[n-m]$
both $x[n]$ and $g[n]$ are causal, thus

$$\begin{aligned} c[n] = \sum_{m=0}^{n} x[m] g[n-m] &= \sum_{m=0}^{n} (0.8)^m u[m] (0.3)^{n-m} u[n-m] \\ &= \begin{cases} \sum_{m=0}^{n} (0.8)^m (0.3)^{n-m} & n \geq 0 \\ 0 & n < 0 \end{cases} \end{aligned}$$

or

$$\begin{aligned} c[n] = (0.3)^n \sum_{m=0}^{n} (0.8/0.3)^m u[n] &= (0.3)^n \frac{1-(0.8/0.3)^{n+1}}{1-(0.8/0.3)} u[n] \\ &= 2\left[(0.8)^{n+1} - (0.3)^{n+1}\right] u[n] \end{aligned}$$

Properties of convolution sum

Commutative

$$x_1[n] * x_2[n] = x_2[n] * x_1[n]$$

Distributive

$$x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n]$$

Associative

$$x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$$

Shifting: if $x_1[n] * x_2[n] = y[n]$ then

$$x_1[n-m] * x_2[n-p] = y[n-m-p]$$

Convolution with an Impulse

$$x[n] * A\delta[n-n_0] = Ax[n-n_0]$$

Differencing: if $c[n] = x_1[n] * x_2[n]$ then

$$c[n] - c[n-1] = x_1[n] * (x_2[n] - x_2[n-1])$$

Summation: if $c[n] = x_1[n] * x_2[n]$ then

$$\text{sum of } c = (\text{sum of } x_1) \times (\text{sum of } x_2)$$

Width and length properties: the *width* of a signal is the number of its elements (length) minus one

- if $x_1[n]$ and $x_2[n]$ have finite widths of W_1 and W_2, respectively, then the width of $x_1[n] * x_2[n]$ is $W_1 + W_2$
- if $x_1[n]$ and $x_2[n]$ have finite lengths of L_1 and L_2 elements, then the length of $x_1[n] * x_2[n]$ is $L_1 + L_2 - 1$ elements

Example 6.8 (convolution from table)

- many convolution sums can be determined from already determined signal pairs (convolution table)
- we can combine these pairs with convolution properties to find more complicated convolutions

Example: use the table to find the following convolutions

(a) $y_a[n] = (0.8)^n u[n] * u[n]$

(b) $y_b[n] = (0.8)^n u[n-1] * u[n+3]$

Solution:

(a) direct application of pair 4 from table gives

$$y_a[n] = (0.8)^n u[n] * u[n] = \frac{0.8^{n+1} - 1}{0.8 - 1} u[n] = 5(1 - (0.8)^{n+1}) u[n]$$

(b) we have

$$y_b[n] = (0.8)^n u[n-1] * u[n+3] = 0.8(0.8)^{n-1} u[n-1] * u[n+3]$$

hence from shifting property

$$y_b[n] = 0.8 y_a[n+2] = 4(1 - (0.8)^{n+3}) u[n+2]$$

Graphical procedure

$$c[n] = x[n] * g[n] = \sum_{m=-\infty}^{\infty} x[m]g[n-m]$$

the convolution operation can be performed as follows:

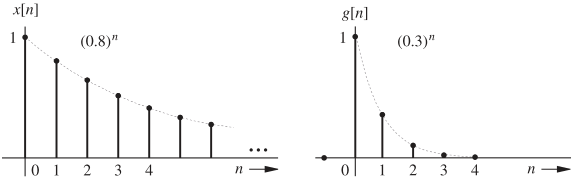
1. we first plot $x[m]$ and $g[n-m]$ as functions of m
2. invert $g[m]$ about the vertical axis $(m = 0)$ to obtain $g[-m]$
3. shift $g[-m]$ by n units to obtain $g[n-m]$
   - for $n > 0$, the shift is to the right (delay)
   - for $n < 0$, the shift is to the left (advance)
4. multiply $x[m]$ and $g[n-m]$ and add all the products to obtain $c[n]$

(the procedure is repeated for each value of n over the range $-\infty$ to ∞)

Example 6.9

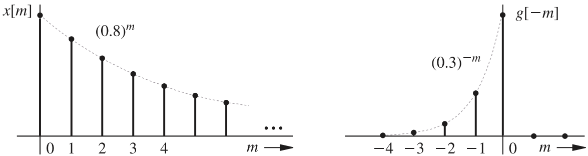
find $c[n] = x[n] * g[n]$, where

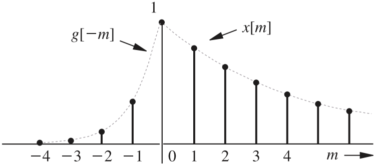
$$x[n] = (0.8)^n u[n] \quad \text{and} \quad g[n] = (0.3)^n u[n]$$

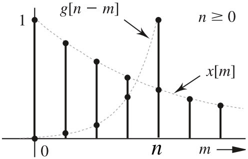


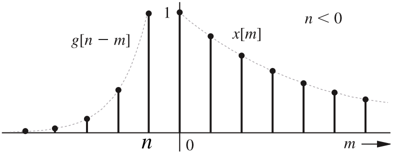
Solution: note that

$$x[m] = (0.8)^m \quad \text{and} \quad g[n-m] = (0.3)^{n-m}$$









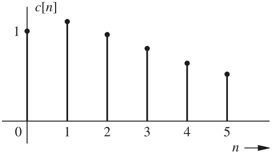
- for $n < 0$, there is no overlap, so that $c[n] = 0$ for $n < 0$
- for $n \geq 0$, the two functions overlap over the interval $0 \leq m \leq n$:

$$c[n] = \sum_{m=0}^{n} x[m]g[n-m] = \sum_{m=0}^{n} (0.8)^m (0.3)^{n-m} = (0.3)^n \sum_{m=0}^{n} \left(\frac{0.8}{0.3}\right)^m$$

$$= 2[(0.8)^{n+1} - (0.3)^{n+1}] \quad n \geq 0$$

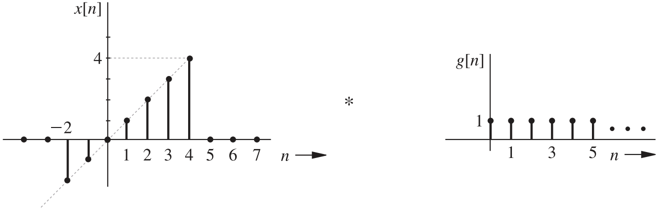
combining pieces, we see that

$$c[n] = 2[(0.8)^{n+1} - (0.3)^{n+1}]u[n]$$

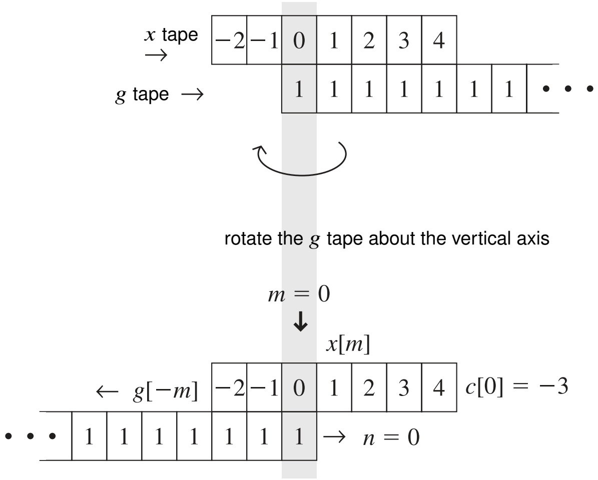


Example 6.10: Sliding-tape method

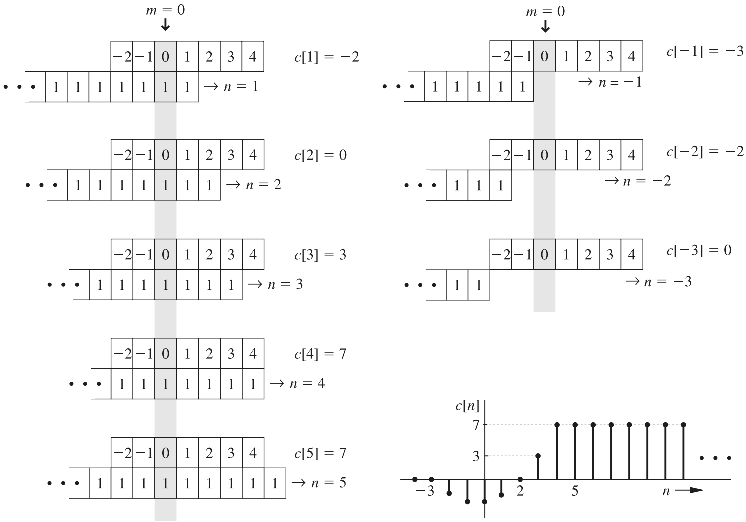
use the sliding-tape method to find $x[n] * g[n]$ for the signals shown below



Solution: in this procedure we represent the sequences $x[m]$ and $g[m]$ as tapes; we then get the $g[-m]$ tape by inverting the $g[m]$ tape about the origin $(m = 0)$



we now shift the inverted tape by n slots, multiply values on two tapes in adjacent slots, and add all the products to find $c[n]$



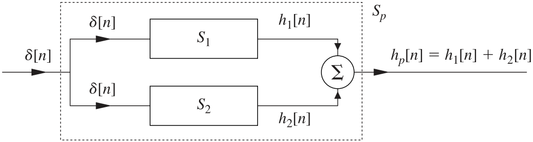
$$c[0] = (-2 \times 1) + (-1 \times 1) + (0 \times 1) = -3$$
$$c[1] = (-2 \times 1) + (-1 \times 1) + (0 \times 1) + (1 \times 1) = -2$$
$$c[2] = (-2 \times 1) + (-1 \times 1) + (0 \times 1) + (1 \times 1) + (2 \times 1) = 0$$
$$c[3] = (-2 \times 1) + (-1 \times 1) + (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) = 3$$
$$c[4] = (-2 \times 1) + (-1 \times 1) + (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) = 7$$
$$c[5] = (-2 \times 1) + (-1 \times 1) + (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) = 7$$
$$c[n] = 7 \quad n \geq 4$$

similarly, we compute $c[n]$ for negative n by sliding the tape backward:

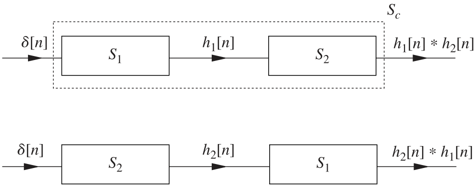
$$c[-1] = (-2 \times 1) + (-1 \times 1) = -3$$
$$c[-2] = (-2 \times 1) = -2$$
$$c[-3] = 0$$
$$c[n] = 0 \quad n \leq -4$$

Interconnected systems

Parallel systems

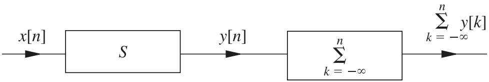


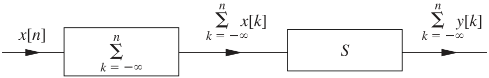
Cascade systems



because $h_1[n] * h_2[n] = h_2[n] * h_1[n]$, linear systems commute; hence, we can interchange the order of cascade systems without affecting the final result

Example:





- if $x[n] \Longrightarrow y[n]$, then $\sum_{k=-\infty}^{n} x[k] \Longrightarrow \sum_{k=-\infty}^{n} y[k]$
- if $x[n] = \delta[n]$ then $y[n] = h[n]$ and $\sum_{k=-\infty}^{n} x[k] = u[n]$

Unit-step response: the unit step response of an LTID system with impulse response $h[n]$ is

$$g[n] = \sum_{k=-\infty}^{n} h[k]$$

it also holds that

$$h[n] = g[n] - g[n-1]$$

Inverse systems

the cascade of a system $h[n]$ with its inverse $h_i[n]$ is an identity system

$$h[n] * h_i[n] = \delta[n]$$

Example: we can show that the accumulator system $y[n] = \sum_{k=-\infty}^{n} x[k]$ and the backward difference system $y[n] = x[n] - x[n-1]$ are the inverse of each other

to see this, note that the impulse response of the accumulator and backward difference systems are is

$$h_{\mathrm{acc}}[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n] \quad \text{and} \quad h_{\mathrm{bdf}}[n] = \delta[n] - \delta[n-1]$$

we can verify that

$$h_{\mathrm{acc}} * h_{\mathrm{bdf}} = u[n] * \{\delta[n] - \delta[n-1]\} = u[n] - u[n-1] = \delta[n]$$

Example 6.11 (total response)

$$\textbf{total response of LTID system} = \text{ZIR} + \underbrace{x[n] * h[n]}_{\text{ZSR}}$$

find the output of the system described by the equation

$$y[n+2] - 0.6y[n+1] - 0.16y[n] = 5x[n+2]$$

with initial conditions $y[-1] = 0$, $y[-2] = 25/4$ and input $x[n] = (4)^{-n}u[n]$

Solution: from slides 6.7 and 6.19 , we know that zero-input response and impulse response are

$$y_0[n] = 0.2(-0.2)^n + 0.8(0.8)^n$$
$$h[n] = [(-0.2)^n + 4(0.8)^n]\, u[n]$$

the zero-state response is:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= (0.25)^n u[n] * \left[(-0.2)^n u[n] + 4(0.8)^n u[n]\right] \\ &= (0.25)^n u[n] * (-0.2)^n u[n] + (0.25)^n u[n] * 4(0.8)^n u[n] \end{aligned}$$

using pair 4 of convolution table, we get

$$\begin{aligned} y[n] &= \left[\frac{(0.25)^{n+1} - (-0.2)^{n+1}}{0.25 - (-0.2)} + 4\frac{(0.25)^{n+1} - (0.8)^{n+1}}{0.25 - 0.8}\right] u[n] \\ &= \left(2.22\left[(0.25)^{n+1} - (-0.2)^{n+1}\right] - 7.27\left[(0.25)^{n+1} - (0.8)^{n+1}\right]\right) u[n] \\ &= \left[-1.26(0.25)^n + 0.444(-0.2)^n + 5.81(0.8)^n\right] u[n] \end{aligned}$$

therefore, the total response for $n \geq 0$ is

$$\text{total response} = \underbrace{0.2(-0.2)^n + 0.8(0.8)^n}_{\text{ZIR}} + \underbrace{0.444(-0.2)^n + 5.81(0.8)^n - 1.26(4)^{-n}}_{\text{ZSR}}$$

Natural and forced response

- when all the characteristic mode terms in the total response are lumped together, the resulting component is the *natural response*
- the remaining part of the total response that is made up of noncharacteristic modes is the *forced response*

Example: the characteristic modes of the previous system are $(-0.2)^n$ and $(0.8)^n$; hence

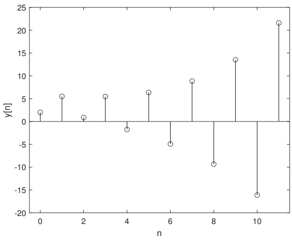
$$\text{total response } = \underbrace{0.644(-0.2)^n + 6.61(0.8)^n}_{\text{natural response}} \underbrace{-1.26(4)^{-n}}_{\text{forced response}} \quad n \geq 0$$

just like differential equations, the classical solution to difference equations includes the natural and forced responses

Finding the zero-state response using MATLAB

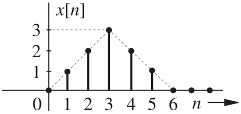
use the MATLAB `filter` command to compute and sketch the zero-state response for the system described by $(E^2 + 0.5E - 1)y[n] = (2E^2 + 6E)x[n]$ with input $x[n] = 4^{-n}u[n]$

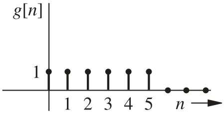
```
n = (0:11); x = @(n) 4.^(-n).*(n>=0);
a = [1 0.5 -1]; b = [2 6 0]; y = filter(b,a,x(n));
clf; stem(n,y,'k'); xlabel('n'); ylabel('y[n]');
axis([-0.5 11.5 -20 25]);
```



although the input is bounded and quickly decays to zero, the system itself is unstable and an unbounded output results.

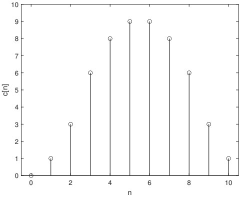
Discrete-time convolution using MATLAB






```
x = [0 1 2 3 2 1]; g = [1 1 1 1 1 1];
n = (0:1:length(x)+length(g)-2);
c = conv(x,g);
clf; stem(n,c,'k'); xlabel('n'); ylabel('c[n]');
axis([-0.5 10.5 0 10]);
```

(the starting point of the result is determined by simply add the starting points of each signal being convolved: in our case $0 + 0 = 0$)



Exercises

- use both the convolution sum definition and table to show that
$(0.8)^n u[n] * u[n] = 5[1 - (0.8)^{n+1}]u[n]$

- use convolution sum table and properties to find
  (a) $n\,3^{-n}u[n] * (0.2)^n u[n]$
  (b) $e^{-n}u[n] * 2^{-n}u[n]$
  (c) $j\delta[n+1] * 2^{-n}u[n]$

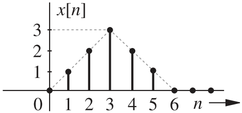
  Answers:
  (a) $\frac{15}{4}[(0.2)^n - (1 - \frac{2}{3}n)3^{-n}]u[n]$; (b) $\frac{2}{2-e}[e^{-n} - \frac{e}{2}2^{-n}]u[n]$

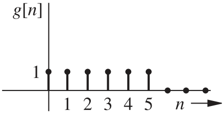
- consider an LTID system that has an impulse response $h[n] = 2^{-n}u[n]$; using the input $x[n] = u[n] - u[n-10]$, determine and sketch the zero-state response of this system over $-10 \le n \le 20$

- find the output (response) $y[n]$ of the system described in example on page 6.43 if the input is modified to be $x[n] = \delta[n] + 4^{-n}u[n]$
Answer: $y[n] = [-1.26(4)^{-n} + 1.444(-0.2)^n + 9.81(0.8)^n]u[n]$

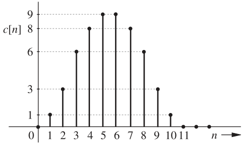
Exercises

- find the convolution $(0.8)^n u[n-1] * u[n+3]$ graphically and sketch the result
  [**Answer:** $4(1-(0.8)^{n+3})u[n+2]$]
- use the sliding-tape technique to find $c[n] = x[n] * g[n]$; also, verify the width property of convolution





Answer:



- use the sliding-tape procedure to determine and plot $y[n] = x[n] * h[n]$ for $x[n] = (3-|n|)(u[n+3]-u[n-4])$ and $h[n] = u[-n+4]-u[-n-2]$; verify the convolution width property

Outline

- zero-input response
- unit-impulse response
- zero-state response and convolution
- **system stability**

BIBO stability

an LTID system is BIBO stable if every bounded input results in a bounded output; or if there exists a K such that

$$\sum_{n=-\infty}^{\infty} |h[n]| < K < \infty$$

- absolutely summable $h[n]$
- otherwise it is unstable

proof: note that

$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} h[m]x[n-m] \right| \le \sum_{m=-\infty}^{\infty} |h[m]||x[n-m]|$$

if $x[n]$ is bounded, then $|x[n-m]| < K_0 < \infty$, and

$$|y[n]| \le K_0 \sum_{m=-\infty}^{\infty} |h[m]|$$

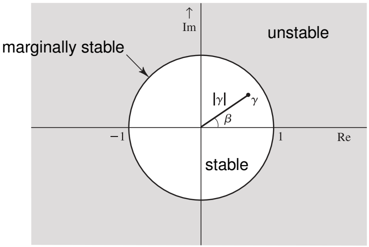
clearly the output is bounded if $\sum_{m=-\infty}^{\infty} |h[m]|$ is bounded

Internal stability

for LTID systems, internal stability, called asymptotical stability or stability in the sense of Lyapunov (also the zero-input stability), is defined in terms of the zero-input response of a system

an LTID system is

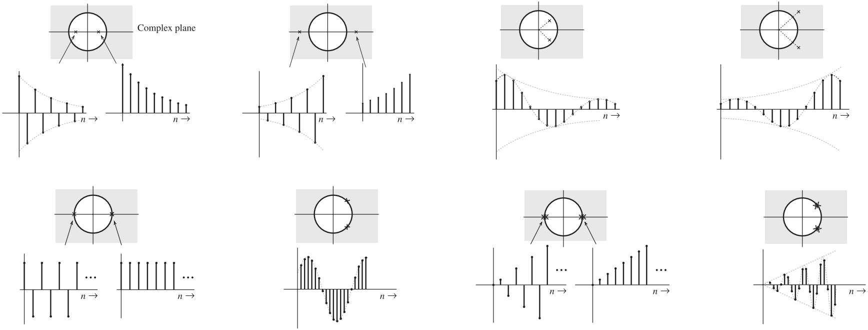
1. *asymptotically stable* if, and only if, all the characteristic roots are inside the unit circle (the roots may be simple or repeated)
2. *marginally stable* if and only if there are no roots outside the unit circle and there are some unrepeated roots on the unit circle
3. *unstable* if, and only if, either one or both of the following conditions exist:
   (i) at least one root is outside the unit circle
   (ii) there are repeated roots on the unit circle



$$\begin{array}{ll} \text{if } |\gamma| < 1, & \text{then } \gamma^n \to 0 \text{ as } n \to \infty \\ \text{if } |\gamma| > 1, & \text{then } \gamma^n \to \infty \text{ as } n \to \infty \\ \text{if } |\gamma| = 1, & \text{then } |\gamma|^n = 1 \text{ for all } n \end{array}$$

Relation with BIBO stability

- an asymptotically stable system is BIBO-stable
- the converse is not necessarily true; BIBO (external) stability cannot ensure internal (asymptotic) stability
- for a difference LTI system, marginal stability or asymptotic instability implies that the system is BIBO-unstable


Complex plane
n →
n →
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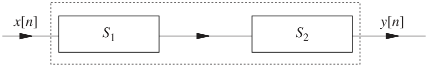
Example 6.12

an LTID systems consists of two subsystems S_1 and S_2 in cascade



the impulse response of these systems are

$$h_1[n] = 4\delta[n] - 3(0.5)^n u[n] \quad \text{and} \quad h_2[n] = 2^n u[n]$$

investigate the BIBO and asymptotic stability of the composite system

Solution: the composite system impulse response $h[n]$ is given by

$$\begin{aligned} h[n] = h_1[n] * h_2[n] = h_2[n] * h_1[n] &= 2^n u[n] * (4\delta[n] - 3(0.5)^n u[n]) \\ &= 4(2)^n u[n] - 3\left[\frac{2^{n+1} - (0.5)^{n+1}}{2 - 0.5}\right] u[n] \\ &= (0.5)^n u[n] \end{aligned}$$

- the system is BIBO-stable because its impulse response $(0.5)^n u[n]$ is absolutely summable
- the system S_2 is asymptotically unstable because its characteristic root, 2, lies outside the unit circle; this system will eventually burn out (or saturate) because of the unbounded characteristic response
- this example shows that BIBO stability does not necessarily ensure asymptotic stability

Example 6.13

determine the internal and external stability of systems specified by the following equations; in each case plot the characteristic roots in the complex plane

(a) $y[n+2] + 2.5y[n+1] + y[n] = x[n+1] - 2x[n]$

(b) $y[n] - y[n-1] + 0.21y[n-2] = 2x[n-1] + 3x[n-2]$

(c) $y[n+3] + 2y[n+2] + \frac{3}{2}y[n+1] + \frac{1}{2}y[n] = x[n+1]$

(d) $\left(E^2 - E + 1\right)^2 y[n] = (3E+1)x[n]$

Solution:

(a) the characteristic polynomial is $\gamma^2 + 2.5\gamma + 1 = (\gamma + 0.5)(\gamma + 2)$ and the characteristic roots are -0.5 and -2; -2 lies outside the unit circle), so the system is BIBO-unstable and also asymptotically unstable

(b) the characteristic polynomial is $\gamma^2 - \gamma + 0.21 = (\gamma - 0.3)(\gamma - 0.7)$ and the characteristic roots are 0.3 and 0.7, both of which lie inside the unit circle; the system is BIBO-stable and asymptotically stable

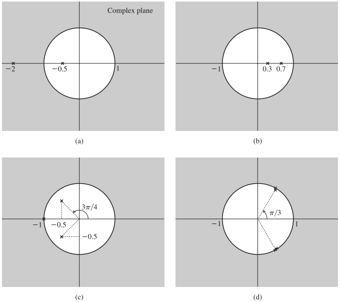
(c) the characteristic polynomial is
$\gamma^3+2\gamma^2+\frac{3}{2}\gamma+\frac{1}{2} = (\gamma+1)\left(\gamma^2+\gamma+\frac{1}{2}\right) = (\gamma+1)(\gamma+0.5-j0.5)(\gamma+0.5+j0.5)$

the characteristic roots are $-1, -0.5 \pm j0.5$; one of the characteristic roots is on the unit circle and the remaining two roots are inside the unit circle; the system is BIBO-unstable but marginally stable

(d) the characteristic polynomial is

$$\left(\gamma^2 - \gamma + 1\right)^2 = \left(\gamma - \tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}\right)^2 \left(\gamma - \tfrac{1}{2} + j\tfrac{\sqrt{3}}{2}\right)^2$$

the characteristic roots are $(1/2) \pm j(\sqrt{3}/2) = 1e^{\pm j(\pi/3)}$ repeated twice, and they lie on the unit circle; the system is BIBO-unstable and asymptotically unstable



Exercise

determine BIBO (external) and asymptotic (internal) stability of each system

(a) $(E+1)\left(E^2+6E+25\right)y[n] = 3Ex[n]$

(b) $(E^2-2E-1)(E+0.5)y[n] = \left(E^2+2E+3\right)x[n]$

Answers: both systems are BIBO-and asymptotically unstable

References

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press, chapter 3.
- M. J. Roberts, *Signals and Systems: Analysis Using Transform Methods and MATLAB*, McGraw Hill, chapter 5 (5.3).