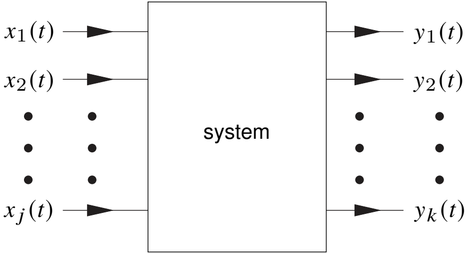
2. Continuous-time systems

- CT systems
- classifications of CT systems
- modeling of basic systems
- introduction to state-space modeling

System

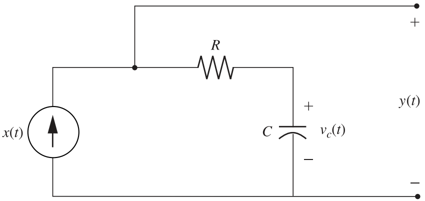
a *system* is an entity that processes input signals to provide output signals

- a system that operates on CT-signals is a *continuous-time systems*
- to excite a system means to apply energy that causes it to respond
- a system can have multiple inputs and multiple outputs (MIMO)



Examples

- *amplifier:* $y(t) = ax(t)$
- *integrator:* $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
- *RC-circuit*



the input current $x(t)$ and output voltage $y(t)$ are related by:

$$y(t) = Rx(t) + v_C(t_0) + \frac{1}{C}\int_{t_0}^{t} x(\tau)d\tau, \quad t \geq t_0$$

System analysis and design

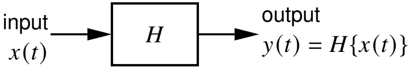
the study of systems consists of three major areas:

- **system modeling:** the mathematical equations relating the outputs to the inputs are called the *system model*
- **system analysis:** how to determine the system outputs for the given inputs and a given mathematical model of the system
- **system design (synthesis):** how to construct a system that will produce a desired set of outputs for the given inputs

Block diagrams

in system analysis it is common and useful to represent systems by *block diagrams*

Single-input single-output

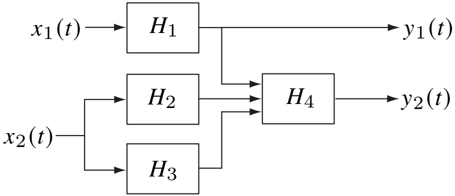


- input $x(t)$ is operated on by the operator H to produce the signal at the output $y(t)$
- the operator H could perform just about any operation imaginable

Interconnected systems

a system is often described and analyzed as an assembly of **components**

- a component is a smaller, simpler system
  - to a circuit designer, components are resistors, capacitors, inductors, operational amplifiers and so on, and systems are power amplifiers, A/D converters, modulators, filters and so forth
  - to an automobile designer components are wheels, engines, bumpers, lights, seats and the system is the automobile

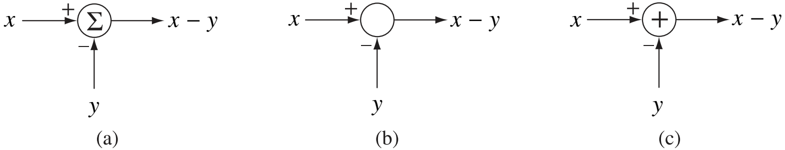


- by knowing the mathematical model of the components, an engineer can predict the behavior (output) of the system

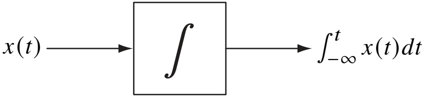
Common block diagram operations

Amplifier (scalar multiplication)

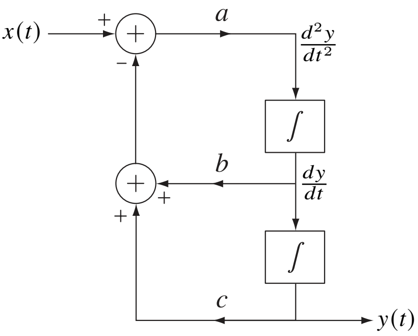
Summation (addition)



Integrator



Example 2.1



$$\frac{dy^2}{dt^2} = a\left(x(t) - \left[b\frac{dy}{dt} + cy(t)\right]\right)$$

or

$$\frac{dy^2}{dt^2} + (ab)\frac{dy}{dt} + (ac)y(t) = ax(t)$$

Outline

- CT systems
- **classifications of CT systems**
- modeling of basic systems
- introduction to state-space modeling

Linear systems

a system H is

- *homogeneous* if $x \rightarrow y$, then $ax \rightarrow ay$ for any number a
- *additive* if $x_1 \rightarrow y_1$, and $x_2 \rightarrow y_2$, then $x_1 + x_2 \rightarrow y_1 + y_2$

Linear systems: a system is *linear* if it is both homogeneous and additive

$$x_1 \longrightarrow y_1$$
$$x_2 \longrightarrow y_2$$

then for any numbers a_1, a_2

$$a_1 x_1 + a_2 x_2 \longrightarrow a_1 y_1 + a_2 y_2$$

the above is called the *superposition property*

Example 2.2

determine whether the following systems are linear or nonlinear

(a) $\dfrac{dy(t)}{dt} + 3y(t) = x(t)$

(b) $y(t)\dfrac{dy(t)}{dt} + 3y(t) = x(t)$

(c) $y(t) = e^{x(t)}$

Solution:

(a) let $y_1(t)$ and $y_2(t)$ to be the outputs for inputs $x_1(t)$ and $x_2(t)$; then,

$$\frac{dy_1(t)}{dt} + 3y_1(t) = x_1(t) \qquad \frac{dy_2(t)}{dt} + 3y_2(t) = x_2(t)$$

multiplying the first equation by a_1 and the second by a_2 and adding, gives

$$\frac{d}{dt}[a_1y_1(t) + a_2y_2(t)] + 3[a_1y_1(t) + a_2y_2(t)] = a_1x_1(t) + a_2x_2(t),$$

which is the system equation with

$$x(t) = a_1x_1(t) + a_2x_2(t), \qquad y(t) = a_1y_1(t) + a_2y_2(t)$$

hence, superposition is satisfied and the system is linear

(b) if $x(t) \longrightarrow y(t)$, then we have

$$y(t)\frac{dy(t)}{dt} + 3y(t) = x(t)$$

multiplying by a, we have

$$ay(t)\frac{dy(t)}{dt} + 3ay(t) = ax(t),$$

which is not equal to

$$ay(t)\frac{d[ay(t)]}{dt} + 3ay(t) = ax(t)$$

hence, the system is nonlinear

(c) for input $ax(t)$, we have $y(t) = e^{ax(t)} \neq ay(t)$

Total response of a linear system

Zero-input response (ZIR)

- ZIR is the output that results only from initial conditions at $t = 0$
- with zero input $x(t) = 0$ for $t \geq 0$

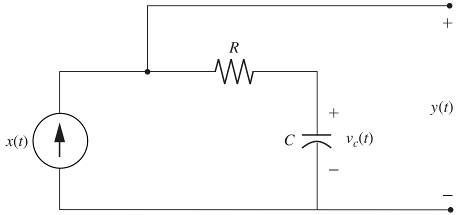
Zero-state response (ZSR)

- ZSR is the output that results from input $x(t)$ for $t \geq 0$
- with zero initial conditions
- when all the initial conditions are zero, the system is said to be in *zero state*

Decomposition property of linear systems:

total response = zero-input response + zero-state response

Example: for the circuit in slide 2.3 (with $t_0 = 0$):



we have

$$y(t) = = \underbrace{v_C(0)}_{\text{ZIR}} + \underbrace{Rx(t) + \frac{1}{C}\int_{t_0}^{t} x(\tau)d\tau,}_{\text{ZSR}} \quad t \geq 0$$

Linearity implication

if we can write $x(t)$ as

$$x(t) = a_1 x_1(t) + a_2 x_2(t) + \cdots + a_m x_m(t)$$

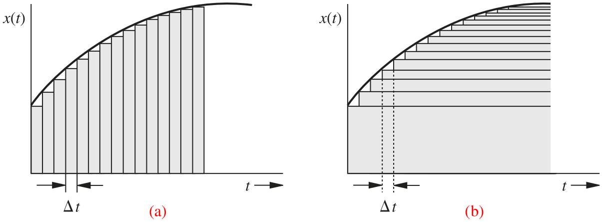
then if the system is linear, the output is

$$y(t) = a_1 y_1(t) + a_2 y_2(t) + \cdots + a_m y_m(t)$$

- $y_k(t)$ is the zero-state response to input $x_k(t)$
- if $x_k(t)$ are simple, then we can find $y(t)$ by finding the responses $y_k(t)$ to the simpler components $x_k(t)$

Linearity implication

any signal can be approximated by a sum of rectangular pulses (impulses) or step-functions



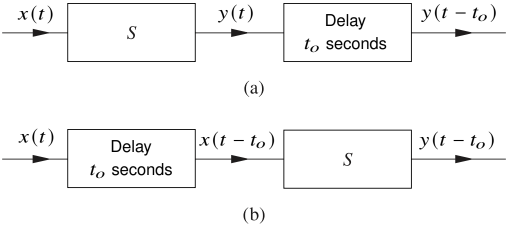
if we know the system response to a unit impulse or unit step input, we can compute the system response to any arbitrary input

Time-invariant systems

a system is *time invariant* if for input-output $x(t) \rightarrow y(t)$, we have

$$x(t - t_o) \rightarrow y(t - t_o)$$

for any arbitrary t_o (assuming initial conditions are also delayed by t_o)



- a system is *time-varying* if the the above does not hold
- a *linear-time-invariant continuous system* (LTIC) is a CT system that both linear and time-invariant

Example 2.3

determine the time invariance of the following systems:

(a) $y(t) = x(t)u(t)$

(b) $y(t) = \frac{d}{dt}x(t)$

(c) $y(t) = e^{-t}x(t)$

(d) $y(t) = e^{x(t)}$

Solution:

(a) the input is modified by a time-dependent function $u(t)$ so the system is time-varying; we can show this through a counterexample:

$$x_1(t) = \delta(t+1) \quad \Longrightarrow \quad y_1(t) = 0$$
$$x_2(t) = x_1(t-2) = \delta(t-1) \quad \Longrightarrow \quad y_2(t) = \delta(t-1)$$

since $y_2(t) \neq y_1(t-2) = 0$, the system is time-varying

(b) for input $x(t)$, we have output $y(t) = \frac{d}{dt}x(t)$; note that

$$y(t-t_o) = \frac{d}{d(t-t_o)}x(t-t_o) = \frac{d}{dt}x(t-t_o),$$

which is the output to a delayed input $x(t-t_o)$; hence, the system is time invariant

(c) the output with delayed input is $e^{-t}x(t-t_o)$, which is not equal to the delayed output $e^{-(t-t_o)}x(t-t_o)$; hence, system is time-varying

(d) let $x_1(t) \to y_1(t) = e^{x_1(t)}$; for input $x_2(t) = x(t-t_o)$ the output is $y_2(t) = e^{x_2(t)} = e^{x(t-t_o)} = y_1(t-t_o)$; hence the system is time invariant

Instantaneous and dynamic systems

Instantaneous (memoryless) system: a system is *instantaneous* (*static*) or *memoryless* if the output at any time t depends, at most, on its input(s) at the same time t, and not on any past or future values of the input(s)

Dynamic systems (with memory)

- a system is *dynamic* or *with memory* if output depends on future or past values of input(s)
- a *finite-memory system with a memory of T seconds* is a system whose output at t is completely determined by the input signals over the past T seconds (interval from $(t - T)$ to t)

Example 2.4

determine whether the following systems are memoryless:

(a) $y(t-1) = 2x(t-1)$

(b) $y(t) = \frac{d}{dt}x(t)$

(c) $y(t) = (t-1)x(t)$

Solution:

(a) memoryless since the output at any time depends on the input at the same time

(b) using the fundamental theorem of calculus:

$$y(t) = \lim_{T \to 0} \frac{x(t) - x(t-T)}{T}$$

the system is not memoryless since the output at a particular time depends on more than just the input at the same time

(c) the system is memoryless since the output at a particular time depends only on the strength of the input at the same time

Causal and noncausal systems

Causal systems: a system is *causal* (*physical* or *nonanticipative*) if the output at t_0 depends only on the input $x(t)$ for $t \leq t_0$

- output does not depend on future input
- the value of output occurs only during or after the time in which it is excited

Noncausal systems: a system that violates the condition of causality is called a *noncausal* (or *anticipative*) system

- unrealizable in *real time* but can be realizable with time delay; for example, we can prerecord data; in such cases, the input's future values are available to us
- noncausal systems are realizable when the independent variable is other than "time" (*e.g.*, space); nontemporal systems, such as those occurring in optics, can be noncausal and still realizable

Example 2.5

determine whether the following systems are causal: (a) $y(t) = x(-t)$, (b) $y(t) = x(t+1)$, (c) $y(t+1) = x(t)$

Solution:

(a) the input $x(t) = \delta(t-1)$, which is nonzero at $t = 1$, produces an output $y(t) = \delta(t+1)$, which is nonzero at $t = -1$, a time 2 seconds earlier than the input; hence, the system is not causal

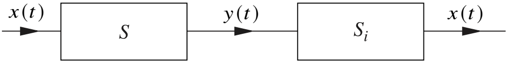
(b) the output at time t depends on the input at future time of $t+1$; thus, the system is not causal

(c) the output at time $t+1$ depends only on the past input (one second in the past); hence, the system is causal

Invertible and noninvertible systems

Invertible systems: a system S is *invertible* if we can find the input $x(t)$ back from the corresponding output $y(t)$ by some operation

- the system that achieves the inverse operation is the *inverse system* for S



- for an invertible system, every input have a unique output (one-to-one mapping between input and output)

Noninvertible systems

- a system is *noninvertible* when it is impossible to obtain the input from the output (several different inputs result in the same output)
- examples: two inputs give same output
  - rectifier: $y(t) = |x(t)|$
  - $y(t) = \sin(x(t))$

Example 2.6

determine whether the following systems are invertible: (a) $y(t) = x(-t)$, (b) $y(t) = tx(t)$, (c) $y(t) = \frac{d}{dt}x(t)$ (d) $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

Solution:

(a) since $x(t) = y(-t)$ for all t, the system is invertible

(b) we have $x(t) = \frac{1}{t}y(t)$ for all t except $t = 0$ and the system is noninvertible since we cannot recover $x(0)$

(c) since the derivative of constants are equal, the system is noninvertible; for example, both $x_1(t) = t + 1$ and $x_2(t) = t - 5$ give the same output

(d) invertible because the input can be obtained by taking the derivative of the output; hence, the inverse system equation is $y(t) = dx/dt$

BIBO stable systems

a system is *bounded-input-bounded-output (BIBO) stable* (*externally stable*) if every bounded input applied at the input terminal results in a bounded output

Examples: determine whether the following systems are BIBO-stable: (i); (ii) $y(t) = tx(t)$; (iii) $y(t) = \frac{d}{dt}x(t)$

(i) the system $y(t) = x^2(t)$ is BIBO stable: if the input is bounded $|x(t)| \leq M_x < \infty$, then $|y(t)| = |x^2(t)| = |x(t)|^2 \leq M_x^2 < \infty$

(ii) the bounded-amplitude input $x(t) = u(t)$ produces the output $y(t) = tu(t)$, which grows to infinity as $t \to \infty$; thus system is a BIBO-unstable system

(iii) the bounded-amplitude input $x(t) = u(t)$ produces the output $y(t) = \delta(t)$ whose amplitude is infinite at $t = 0$; thus, the system is a BIBO-unstable

Linear differential system

$$a_0 \frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_N y(t)$$
$$= b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt} + \cdots + b_M x(t)$$

- order is N
- the system described by differential equation of the above form is linear
- the system is time-invariant if a_i, b_i are constants (independent of time)
- many practical systems can be modeled/approximated by linear differential systems
- without loss of generality, we assume that $a_0 = 1$ since if not, then we can always divide both sides by a_0

Differentiation notations

- the are several notation for differentiation:

$$\dot{y}(t) = y'(t) := \frac{dy(t)}{dt}, \quad \ddot{y}(t) = y''(t) := \frac{d^2y(t)}{dt^2}, \quad \ldots, \quad y^{(N)} := \frac{d^N y(t)}{dt^N}$$

- for convenience , we often use D instead of d/dt:

$$\frac{dy(t)}{dt} := Dy(t), \quad \frac{d^2y(t)}{dt^2} := D^2y(t), \quad \ldots, \quad \frac{d^N y(t)}{dt^N} := D^N y(t)$$

- using the above, the linear differential system becomes

$$(a_0 D^N + a_1 D^{N-1} + \cdots + a_N)y(t) = (b_0 D^M + b_{M-1} D^1 + \cdots + b_M)x(t)$$

Integration operation

$$\int_{-\infty}^{t} y(\tau)d\tau := \frac{1}{D}y(t)$$

Exercises

- show that the system with the input $x(t)$ and the output $y(t)$ described by $y(t) = \mathrm{Re}\{x(t)\}$ satisfies the additivity property but violates the homogeneity property (hence, nonlinear)
- determine whether that the system described by $y(t) = (\sin t)x(t-2)$ is time-invariant or time-varying
- determine whether the the system whose input-output relationship is $y(t) = x(t/2)$ time invariant or not
- show that a system described by the following equation is noncausal

$$y(t) = \int_{t-5}^{t+5} x(\tau)d\tau$$

show that this system can be realized physically if we accept a delay of 5 seconds in the output

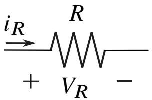
Outline

- CT systems
- classifications of CT systems
- **modeling of basic systems**
- introduction to state-space modeling

Basic electrical elements laws

Resistor

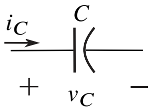
$$v_R = i_R R$$



Capacitor

$$i_C = C\frac{dv_C}{dt}$$

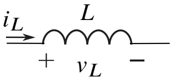
$$v_C(t) = \frac{1}{C}\int_{t_0}^{t} i_C \, d\tau + v_C(t_0)$$



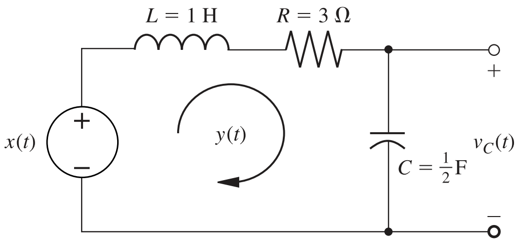
Inductor

$$v_L = L\frac{di_L}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{t_0}^{t} v_L \, d\tau + i(t_0)$$



Example 2.7



find the input-output equation relating the input voltage $x(t)$ to the output current (loop current) $y(t)$

Solution: KVL, gives

$$v_L(t) + v_R(t) + v_C(t) = x(t)$$

using voltage current-law for each element we obtain:

$$\frac{dy(t)}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

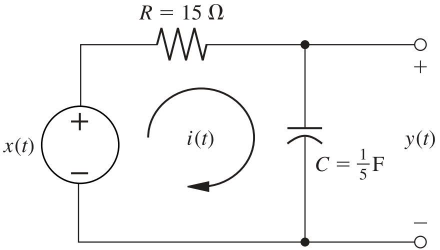
differentiating both sides, we get the input-output relation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

We can write the above as

$$(D + 3D + 2)y(t) = Dx(t)$$

Example 2.8



find the equation relating input-output if the input is the voltage $x(t)$ and output is

(a) the loop current $i(t)$

(b) the capacitor voltage $y(t)$

Solution:

(a) the loop equation is

$$15i(t) + 5\int_{-\infty}^{t} i(\tau)d\tau = x(t)$$

in operator notation, we have

$$15i(t) + \frac{5}{D}i(t) = x(t)$$

multiplying both sides by D (*i.e.*, differentiating the equation), we obtain

$$(15D + 5)i(t) = Dx(t)$$

(b) using $i(t) = C\frac{dy(t)}{dt} = \frac{1}{5}Dy(t)$, we get

$$(3D + 1)y(t) = x(t)$$

Exercises

- If the inductor voltage $v_L(t)$ is taken as the output, show that the RLC circuit in page 2.31 has an input-output equation of

$$(D^2 + 3D + 2)v_L(t) = D^2 x(t)$$

- If the capacitor voltage $v_C(t)$ is taken as the output, show that the RLC circuit in page 2.33 has an input-output equation of

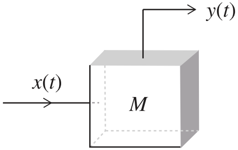
$$(D^2 + 3D + 2)v_C(t) = 2x(t)$$

Mechanical translational laws

the basic elements used in modeling translational systems (moving along a straight line) are ideal masses, linear springs, and dashpots providing viscous damping

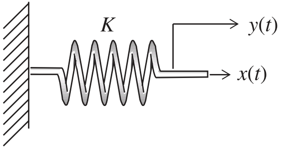
Newton's law of motion: a force $x(t)$ on *mass* M causes a motion $y(t)$ and acceleration $\ddot{y}(t)$

$$x(t) = M\ddot{y}(t) = MD^2y(t)$$



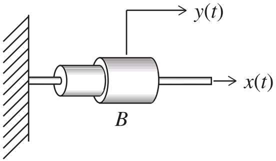
Linear spring: force $x(t)$ required to stretch (or compress) a linear spring with *stiffness* K by amount $y(t)$

$$x(t) = Ky(t)$$



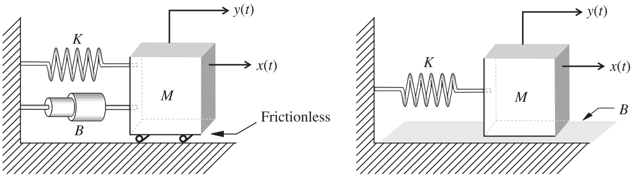
Linear dashpot: the force $x(t)$ moving the dashpot with *damping coefficient* B is proportional to the relative velocity $\dot{y}(t)$ of one surface with respect to the other

$$x(t) = B\dot{y}(t) = BDy(t)$$

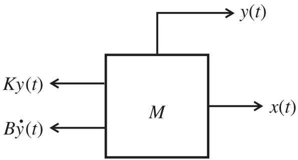


Example 2.9

find the input-output relationship for the translational mechanical system shown below; the input is the force $x(t)$, and the output is the mass position $y(t)$



Solution: in mechanical systems it is helpful to draw a free-body diagram of each junction, which is a point at which two or more elements are connected



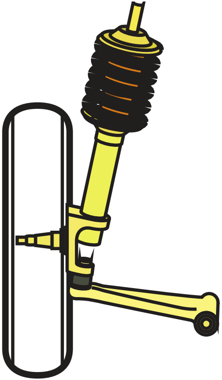
from Newton's second law, the net force must be

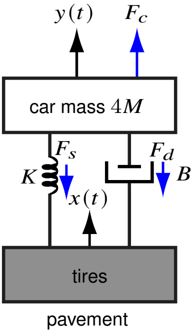
$$M\ddot{y}(t) = -B\dot{y}(t) - Ky(t) + x(t)$$

or

$$(MD^2 + BD + K)y(t) = x(t)$$

Example 2.10 (car suspension system)





- the input $x(t)$ is the vertical displacement of the pavement, defined relative to a reference ground level
- the output $y(t)$ is the vertical displacement of the car chassis from its equilibrium position
- M is one-fourth of the car's mass, because the car has four wheels

- forces exerted by the spring F_s and shock absorber F_d depend on the relative displacement $(y - x)$ of the car relative to the pavement
- when $(y - x)$ is positive (car mass moving away from the pavement), the spring force F_s is directed downward; hence, $F_s = -K(y - x)$
- similarly, the damper force F_d exerted by the shock absorber also is pointed downward: $F_d = -B\dfrac{d}{dt}(y - x)$
- they act to oppose the upward inertial force F_c on the car, which depends on only the car displacement $y(t)$
- using Newton's law, $F_c = Ma = M\left(d^2y/dt^2\right)$, the force equation is $F_c = F_s + F_d$ or

$$M\frac{d^2y}{dt^2} = -K(y - x) - B\frac{d}{dt}(y - x)$$

which can be written as

$$\frac{d^2y}{dt^2} + \frac{B}{M}\frac{dy}{dt} + \frac{K}{M}y = \frac{B}{M}\frac{dx}{dt} + \frac{K}{M}x$$

this is a second-order linear differential system

Outline

- CT systems
- classifications of CT systems
- modeling of basic systems
- **introduction to state-space modeling**

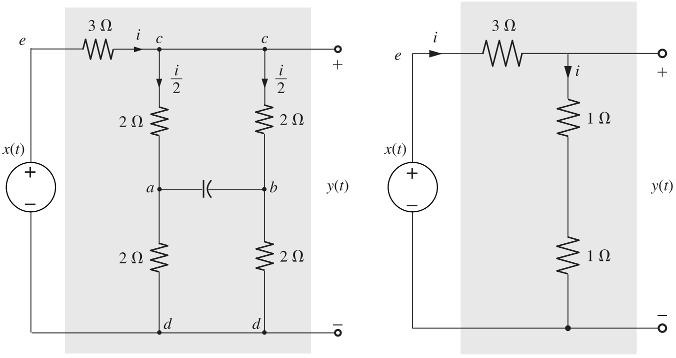
External and internal descriptions

External description: a description that can be obtained from measurements at the external terminals is an *external description*

Internal description: an *internal description* is capable of providing complete information about all possible signals in the system

- the input-output description is an external description
- external descriptions may not provide complete information about all signals in the systems
- an external description can always be found from an internal description, but the converse is not necessarily true

Example: external description



(a) (b)

assume that there is some initial charge Q_0 present on the capacitor: the output $y(t)$ will depend on input $x(t)$ and initial charge Q_0

- *zero-input response:* when $x(t) = 0$ (short input), the currents in the two 2Ω resistors in the upper and the lower branches at the output terminals are equal and opposite because of the balanced nature of the circuit; hence, $y(t) = 0$ even if there is initial charge

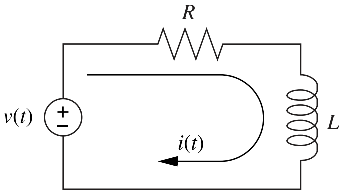
- *zero-state response:* when $Q_0 = 0$ (short capacitor), then the current divides equally between branches and the voltage across the capacitor continues to remain zero and we get the equivalent circuit on right

$$y(t) = 2i(t) = \frac{2}{5}x(t)$$

- the total response $y(t) = \frac{2}{5}x(t)$, which gives the external description; no external measurement can detect the presence of the capacitor

Example: state-space internal description

consider the RL network shown with an initial current of $i(0)$



1. we select the current, $i(t)$, for which we will write and solve a differential equation using Laplace transforms
2. we write the loop equation,

$$L\frac{di}{dt} + Ri = v(t)$$

3. solving gives: $i(t) = \frac{1}{R}\left(1 - e^{-(R/L)t}\right) + i(0)e^{-(R/L)t}$

4. we can now solve for all of the other network variables algebraically in terms of $i(t)$ and the input voltage:

$$v_R(t) = Ri(t) \quad \text{(output equation 1)}$$
$$v_L(t) = v(t) - Ri(t) \quad \text{(output equation 2)}$$

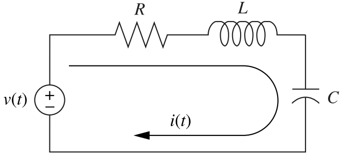
knowing the state variable, $i(t)$, and the input, $v(t)$, we can find the value, or state, of any network variable at any time, $t \geq t_0$

5. since the variables of interest are completely described by the last three equations, we say that these combined equations form a viable representation of the network, which we call a state-space representation

$$\frac{di}{dt} = \frac{1}{L}[v(t) - Ri(t)] \quad \text{(state equation)}$$
$$v_R(t) = Ri(t) \quad \text{(output equation 1)}$$
$$v_L(t) = v(t) - Ri(t) \quad \text{(output equation 2)}$$

Example: Second-order system

consider the second-order circuit



1. since the network is of second order, two simultaneous, first-order differential equations are needed to solve for two state variables; we select $i(t)$ and $q(t)$, the charge on the capacitor, as the two state variables
2. writing the loop equation yields

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t)$$

converting to charge, using $i(t) = dq/dt$, we get

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v(t)$$

we can convert the previous equation into two simultaneous, first-order differential equations in terms of $i(t)$ and $q(t)$

the first equation can be $dq/dt = i$; the second equation can be formed by substituting $\int i dt = q$ into the equation and solving for di/dt:

$$\frac{dq}{dt} = i$$
$$\frac{di}{dt} = -\frac{1}{LC}q - \frac{R}{L}i + \frac{1}{L}v(t)$$

3. these equations are the state equations and can be solved simultaneously for the state variables, $q(t)$ and $i(t)$ if we know the input, $v(t)$, and the initial conditions for $q(t)$ and $i(t)$

4. from these two state variables, we can solve for all other network variables; for example:

$$v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t)$$

this equation is an output equation

5. a state-space representation of the circuit is:

$$\begin{aligned} \frac{dq}{dt} &= i \\ \frac{di}{dt} &= -\frac{1}{LC}q - \frac{R}{L}i + \frac{1}{L}v(t) \\ v_L(t) &= -\frac{1}{C}q(t) - Ri(t) + v(t) \end{aligned}$$

Vector matrix form: the state equations, can be written as

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}x$$

where

$$\dot{\mathbf{q}} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}$$
$$\mathbf{q} = \begin{bmatrix} q \\ i \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}; \quad x = v(t)$$

the output equation, can be written as

$$y = \mathbf{C}\mathbf{x} + Dx$$

where

$$y = v_L(t); \quad \mathbf{C} = \begin{bmatrix} -1/C & -R \end{bmatrix}; \quad \mathbf{q} = \begin{bmatrix} q \\ i \end{bmatrix}; \quad D = 1; \quad x = v(t)$$

Uniqueness: a state-space representation is not unique, since a different choice of state variables leads to a different representation of the same system

Example: if we choose $v_R(t)$ and $v_C(t)$ to be the state variables in the previous example, then state equation become:

$$\frac{dv_R}{dt} = -\frac{R}{L}v_R - \frac{R}{L}v_C + \frac{R}{L}v(t)$$
$$\frac{dv_C}{dt} = \frac{1}{RC}v_R$$

State-space description

State-space modeling: the *state-space model* of a system is an internal description where all signals in the system are expressed using state variables

- **System variable:** any variable that responds to an input or initial conditions in a system
- **State variables:** the minimal number of variables $q_1(t), q_2(t), \ldots, q_N(t)$ such that their value at t_0 are sufficient to determine all signals in the system for $t \geq t_0$ (given the input(s) for $t \geq t_0$) are called *state variables*
- **State vector:** a vector whose elements are the state variables
- **State equations:** a set of n simultaneous, first-order differential equations relating state variables
- **Output equation:** the algebraic equation that expresses the output variables of a system as linear combinations of the state variables and the inputs

(for input-output description, an Nth-order differential system is described by an Nth-order equation; in the state-variable approach, the same system can be described by N simultaneous first-order state equations)

Linear system state-space equations

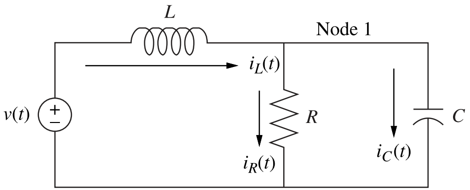
for a linear continuous-time systems, the state and output equations can be expressed as:

$$\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{q} + \boldsymbol{B}\boldsymbol{x}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{q} + \boldsymbol{D}\boldsymbol{x}$$

- $\boldsymbol{q} = (q_1, q_2, \ldots, q_N)$ is the *state vector*
- $\boldsymbol{x} = (x_1, x_2, \ldots, x_M)$ is the *input vector*
- $\boldsymbol{A}$ is an $N \times N$ matrix and $\boldsymbol{B}$ is an $N \times M$ matrix
- $\boldsymbol{y} = (y_1, y_2, \ldots, y_k)$ is the *output vector*
- $\boldsymbol{C}$ is an $k \times N$ matrix and $\boldsymbol{D}$ is an $k \times M$ matrix

Example 2.11

given the electrical network, find a state-space representation if the output is the current through the resistor



Solution: the following steps will yield a viable representation of the network in state-space

- select the state variables by writing the derivative equation for all energystorage elements, that is, the inductor and the capacitor:

$$C\frac{dv_C}{dt} = i_C$$
$$L\frac{di_L}{dt} = v_L$$

  we choose the state variables as the quantities that are differentiated, namely v_C and i_L

- since i_C and v_L are not state variables, our next step is to express i_C and v_L as linear combinations of the state variables, v_C and i_L, and the input, $v(t)$

- we now use Kirchhoff's voltage and current laws, to obtain i_C and v_L in terms of the state variables, v_C and i_L; at Node 1,

$$\begin{aligned} i_C &= -i_R + i_L \\ &= -\frac{1}{R} v_C + i_L \end{aligned}$$

which yields i_C in terms of the state variables, v_C and i_L

- around the outer loop,

$$v_L = -v_C + v(t)$$

which yields v_L in terms of the state variable, v_C, and the source, $v(t)$

- putting things together

$$C\frac{dv_C}{dt} = -\frac{1}{R}v_C + i_L$$
$$L\frac{di_L}{dt} = -v_C + v(t)$$

or

$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L$$
$$\frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}v(t)$$

- we now find the output equation; since the output is $i_R(t)$,

$$i_R = \frac{1}{R}v_C$$

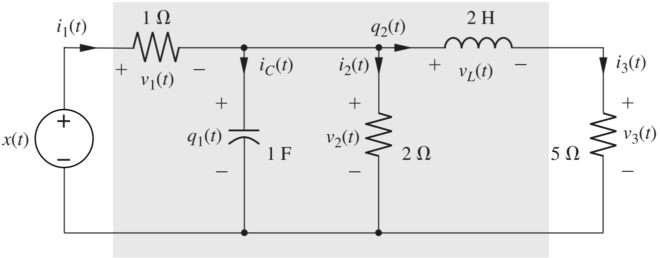
the final result for the state-space representation is found by representing them in vector-matrix form as follows:

$$\begin{bmatrix} v_C' \\ i_L' \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t)$$

$$i_R = \begin{bmatrix} 1/R & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

where $(.)'$ indicates differentiation with respect to time

Example 2.12



find the state equations using the state variables $q_1(t)$ (capacitor voltage) and the $q_2(t)$ (inductor current); verify that all possible system signals at some instant t can be determined from the system state and the input at t

Solution: $\dot{q}_1$ is the current through the capacitor and $2\dot{q}_2$ is the voltage across the inductor; using KCL and KVL, we have

$$\dot{q}_1 = i_C = i_1 - i_2 - q_2 = (x - q_1) - 0.5q_1 - q_2 = -1.5q_1 - q_2 + x$$
$$2\dot{q}_2 = q_1 - v_3 = q_1 - 5q_2 \quad \Longleftrightarrow \quad \dot{q}_2 = 0.5q_1 - 2.5q_2$$

thus, the state equations are

$$\dot{q}_1 = -1.5q_1 - q_2 + x$$
$$\dot{q}_2 = 0.5q_1 - 2.5q_2$$

in matrix form, we have

$$\underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{\boldsymbol{q}}} = \underbrace{\begin{bmatrix} -1.5 & -1 \\ 0.5 & -2.5 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\boldsymbol{q}} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\boldsymbol{B}} x$$

once we solve for q_1 and q_2 at t, we can use KCL and KVL to find any possible signal (current/voltage) in the circuit at t:

$$\begin{aligned} i_1 &= (x - q_1)/1 & i_C &= (x - q_1)/1 - q_1/2 - q_2 \\ v_1 &= x - q_1 & i_3 &= q_2 \\ v_2 &= q_1 & v_3 &= 5q_2 \\ i_2 &= q_1/2 & v_L &= q_1 - 5q_2 \end{aligned}$$

if the outputs are $y_1 = v_1$ and $y_2 = i_C$, then the output equations are

$$\begin{aligned} y_1 &= x - q_1 \\ y_2 &= -(3/2)q_1 - q_2 + x \end{aligned}$$

or in matrix form:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{bmatrix} -1 & 0 \\ -3/2 & -1 \end{bmatrix}}_{\boldsymbol{C}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\boldsymbol{q}} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\boldsymbol{D}} x$$

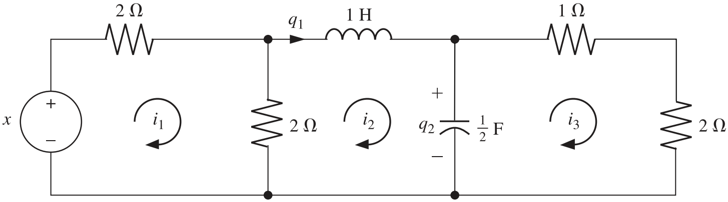
Finding state equations of electrical circuits

Mesh-current method procedure

1. choose all independent capacitor voltages and inductor currents to be the state variables
2. write the mesh-loop currents equations and express the state variables and their first derivatives in terms of the loop currents
3. eliminate all variables other than state variables (and their first derivatives)

Example 2.13

write the state equations for the circuit shown



Solution: there is one inductor and one capacitor in the network; thus, we choose the inductor current q_1 and the capacitor voltage q_2 as the state variables

loop equations:

$$
\begin{aligned}
4i_1 - 2i_2 &= x \\
2\left(i_2 - i_1\right) + \dot{q}_1 + q_2 &= 0 \\
-q_2 + 3i_3 &= 0
\end{aligned}
$$

loop currents and state variables relation:

$$
\begin{aligned}
q_1 &= i_2 \\
\frac{1}{2}\dot{q}_2 &= i_2 - i_3
\end{aligned}
$$

from the second loop equation, we have

$$\dot{q}_1 = 2(i_1 - i_2) - q_2 = -i_2 + 2i_1 - i_2 - q_2$$

using $q_1 = i_2$ and $2i_1 - i_2 = 1/2x$, we can eliminate i_1 and i_2, to obtain

$$\dot{q}_1 = -q_1 - q_2 + \frac{1}{2}x$$

using $q_1 = i_2$ and $\frac{1}{2}\dot{q}_2 = i_2 - i_3$ and the last loop equation, we get

$$\dot{q}_2 = 2q_1 - \frac{2}{3}q_2$$

hence the state equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} x$$

Advantages time-domain analysis

- with the arrival of space exploration, requirements for systems increased in scope; modeling systems by using linear, time-invariant differential equations and subsequent transfer functions became inadequate
- the state-space approach (also referred to as the modern, or time-domain, approach) is a unified method for modeling, analyzing, and designing a wide range of systems
- for example, the state-space approach can be used to represent nonlinear and time-varying systems (e.g., missiles with varying fuel levels or lift in an aircraft flying through a wide range of altitudes)
- multiple-input, multiple-output systems (such as a vehicle with input direction and input velocity yielding an output direction and an output velocity) can be compactly represented in state-space with a model similar in form and complexity to that used for single-input, single-output systems
- the time-domain approach can be used to represent systems with a digital computer in the loop or to model systems for digital simulation; with a simulated system, system response can be obtained for changes in system parameters-an important design tool

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- M. J. Roberts, *Signals and Systems: Analysis Using Transform Methods and MATLAB*, McGraw Hill, chapter 4 (4.1–4.2)