1. Continuous-time signals

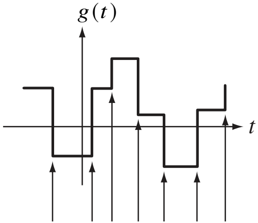
- continuous-time signals
- signal operations
- useful CT signals
- even and odd signals
- signal energy and power

Continuous-time signal

a *continuous-time (CT) signal* is a function $x(t)$ defined at every time t

- voltage, current, audio signals
- light intensity variations in an optical fiber
- position or velocity of moving object

a continuous-time function is not the same as continuous function



point of discontinuity of $g(t)$

Sinusoids and exponentials

Sinusoids

$$x(t) = A\cos(2\pi f t + \theta)$$

- f is the (cyclic) *frequency* (in Hertz); $T = 1/f$ is the *period*
- A is the *amplitude* and θ is the *phase* (in degrees or radians):
- $\omega = 2\pi f = 2\pi/T$ is the *radian frequency*

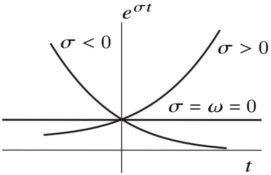
Exponentials

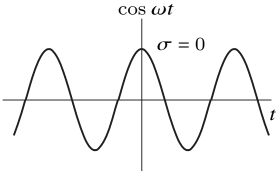
$$x(t) = Ae^{st} = Ae^{(\sigma + j\omega)t} = Ae^{\sigma t}\left(\cos\omega t + j\sin\omega t\right)$$

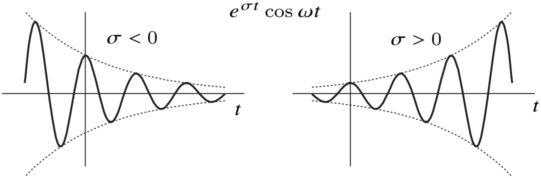
- $s = \sigma + j\omega$ is called *complex frequency*
- $|\omega|$ is called *radian frequency* or frequency of oscillation
- σ is the *decay rate* or *neper frequency*

several functions can be expressed in terms of e^{st}:

- constant: $k = ke^{0t}$ $(s = 0)$
- monotonic exponential: $e^{\sigma t}$ $(\omega = 0)$
- sinusoid: $\cos \omega t = \mathrm{Re}(e^{\pm j\omega t})$ $(\sigma = 0,\ \omega = \pm j\omega)$
- exponentially varying sinusoid: $e^{\sigma t} \cos \omega t$ $(s = \sigma \pm j\omega)$





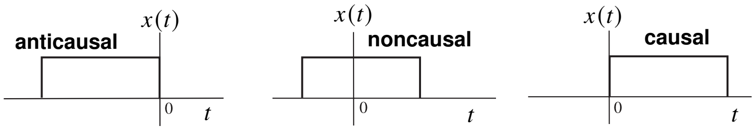


Causal signals

a signal $x(t)$ is *causal* if

$$x(t) = 0 \quad \text{for } t < 0$$

- causal signals do not start before $t = 0$
- a signal that starts before $t = 0$ is called *noncausal*
- a signal $x(t)$ is *anticausal* if $x(t) = 0, \ t \geq 0$
- a signal that exists over $-\infty < t < \infty$ is called *everlasting signal*



Periodic and aperiodic signals

a signal $x(t)$ is *periodic* if for some positive constant T

$$x(t) = x(t+T), \quad \text{for all } t$$

- smallest T is called *fundamental period* of $x(t)$, denoted by T_0
- $f_0 = 1/T_0$ is *cyclic frequency*; $\omega_0 = 2\pi f_0$ is *radian frequency*
- a periodic signal must be an everlasting signal
- areas under $x(t)$ over any interval of duration T_0 are equal

$$\int_a^{a+T_0} x(t)\, dt = \int_b^{b+T_0} x(t)\, dt \triangleq \int_{T_0} x(t)\, dt$$

- a signal is *aperiodic* if it is not periodic

Sum of periodic signals

$$x(t) = x_1(t) + x_2(t)$$

- $x_1(t)$ and $x_2(t)$ are is periodic with periods T_{01} and T_{02}
- $x(t)$ is periodic with period T if there exists a time T that is an integer multiple of both T_{01} and T_{02}:

$$qT_{01} = pT_{02}$$

Fundamental period: the fundamental period T_0 of $x(t)$ is the *least common multiple* (LCM) of T_{01}, T_{02}

- if T_{01}/T_{02} is a rational number, then $x(t)$ is periodic; otherwise, it is aperiodic
- if $T_{01}/T_{02} = p_0/q_0$ for some integers p_0 and q_0 in **smallest form**, then

$$T_0 = \text{LCM}(T_{01}, T_{02}) = q_0 T_{01} = p_0 T_{02}$$

Example 1.1

- the function $x(t) = 3 + t^2$ is aperiodic
- the function $x(t) = e^{-j60\pi t}$ can be expressed as

$$x(t) = \cos(60\pi t) - j\sin(60\pi t)$$

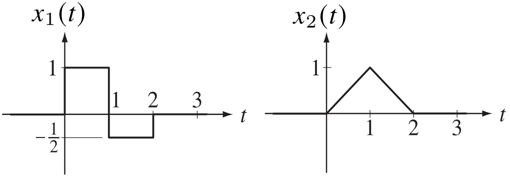
  which is a sum of two periodic signals that have the same fundamental period $T_{01} = T_{02} = 2\pi/60\pi = 1/30$; thus, the fundamental period $T_0 = 1/30\ \mathrm{s}$

- the function $x(t) = 10\sin(12\pi t) + 4\cos(18\pi t)$ is the sum of two periodic functions with $T_{01} = 1/6$ second and $T_{02} = 1/9$ second; we have $T_{01}/T_{02} = 9/6 = 3/2$ and the $T_0 =$ LCM$(1/6, 1/9) = 1/3$
- the function $x(t) = 10\sin(12\pi t) + 4\cos(18t)$ is the sum of two periodic functions with $T_{01} = 1/6$ second and $T_{02} = \pi/9$ seconds; the ratio of the two fundamental periods is $2\pi/3$ irrational; therefore $x(t)$ is aperiodic

Piecewise signals

a *piecewise signal* is a function with different expressions over different intervals

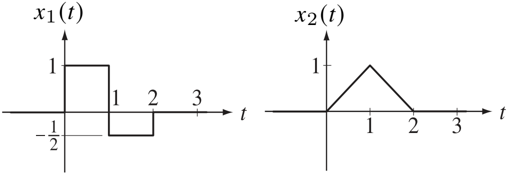
Example



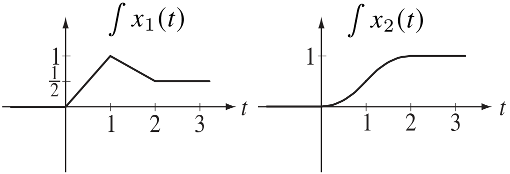
$$x_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1/2 & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad x_2(t) = \begin{cases} t & 0 \le t \le 1 \\ -(t-2) & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Exercises

find and sketch $\int_{-\infty}^{t} x_1(\tau)d\tau$ and $\int_{-\infty}^{t} x_2(\tau)d\tau$ for the signals $x_1(t)$ and $x_2(t)$



Answer:



Outline

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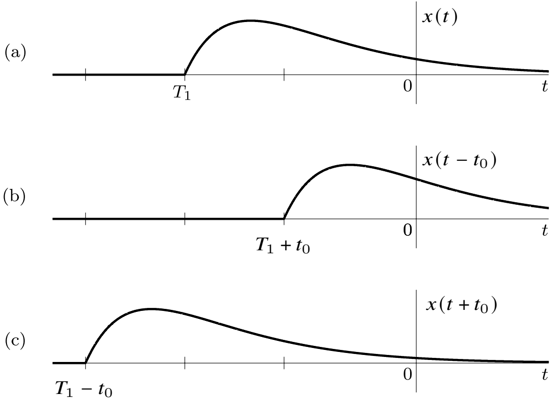
Time shifting

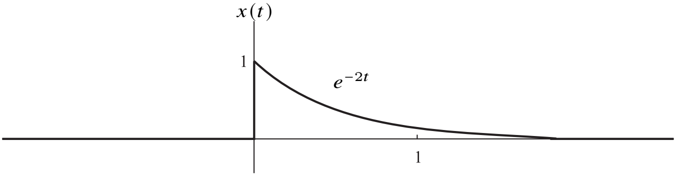
$x(t)$ can be shifted to the right or left by $t_0 > 0$ seconds:

$$x(t - t_0) \quad \text{(right-shifted (delayed) signal)}$$

$$x(t + t_0) \quad \text{(left-shifted (advanced) signal)}$$



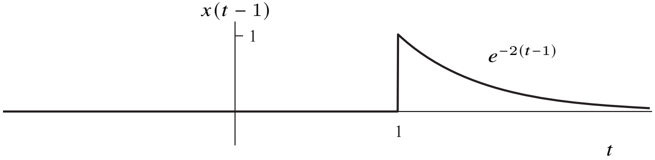
Example 1.2

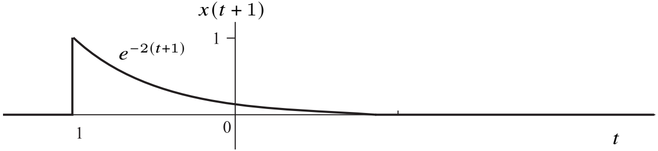


$$x(t) = \begin{cases} e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

sketch and mathematically describe the function $x(t)$ delayed by 1 second and advanced by 1 second

Solution:





$$x(t-1) = \begin{cases} e^{-2(t-1)} & t-1 \geq 0 \\ 0 & t-1 < 0 \end{cases}$$

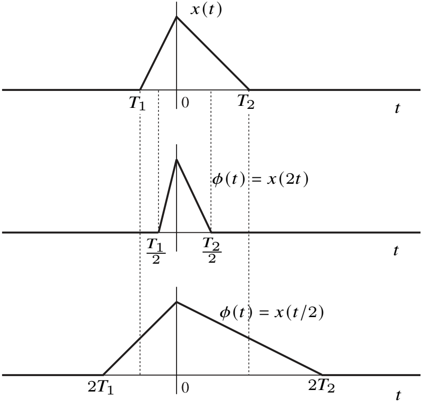
$$x(t+1) = \begin{cases} e^{-2(t+1)} & t+1 \geq 0 \\ 0 & t+1 < 0 \end{cases}$$

Time scaling

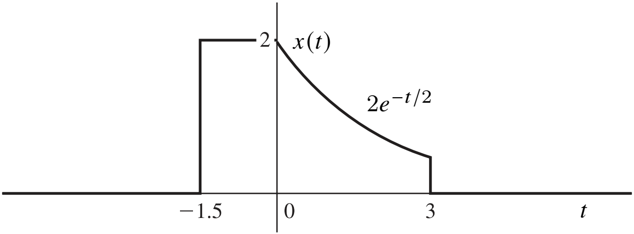
time scaling is the compression or expansion of a signal in time:

$$\phi(t) = x(\alpha t), \quad \text{compression by factor } \alpha > 1$$

$$\phi(t) = x(t/\alpha), \quad \text{expansion by factor } \alpha > 1$$



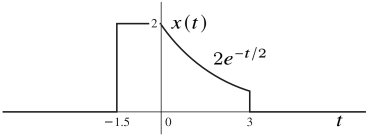
Example 1.3



sketch and mathematically describe the signal $x(t)$ time-compressed by factor 3; repeat the problem for the same signal time-expanded by factor 2

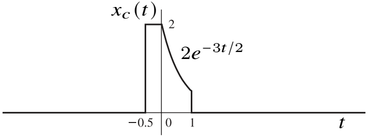
Solution:

$$x(t) = \begin{cases} 2 & -1.5 \le t < 0 \\ 2e^{-t/2} & 0 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$$



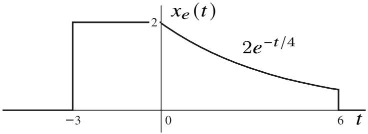
compressed-signal

$$x_c(t) = x(3t) = \begin{cases} 2 & -1.5 \le 3t < 0 \\ 2e^{-3t/2} & 0 \le 3t < 3 \\ 0 & \text{otherwise} \end{cases}$$



expanded signal

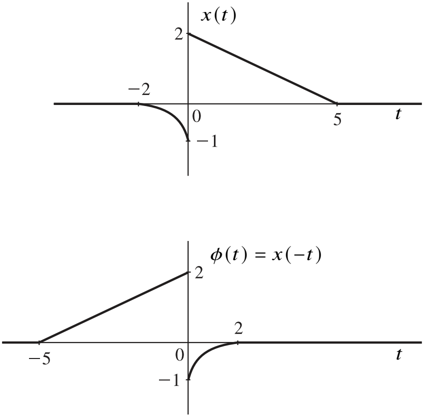
$$x_e(t) = x(t/2) = \begin{cases} 2 & -1.5 \le t/2 < 0 \\ 2e^{-t/4} & 0 \le t/2 < 3 \\ 0 & \text{otherwise} \end{cases}$$



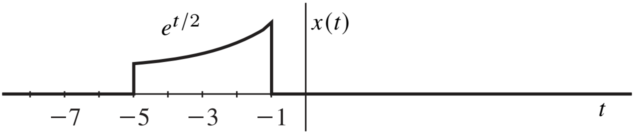
Time reversal

time-reversal is the reflection about the vertical axis

$$\phi(t) = x(-t)$$

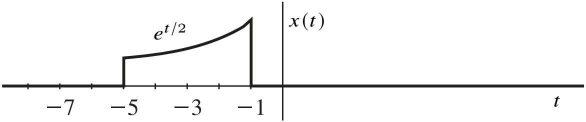


Example 1.4

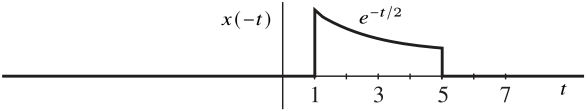


sketch and mathematically describe $x(-t)$

Solution:



(a)



$$x(t) = \begin{cases} e^{t/2} & -5 < t \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

$$x(-t) = \begin{cases} e^{-t/2} & -5 < -t \leq -1 \ (1 \leq t < 5) \\ 0 & \text{otherwise} \end{cases}$$

Combined operations

$$x(\alpha t - t_0) = x\left(\alpha\left[t - \frac{t_0}{\alpha}\right]\right)$$

1. time shift, then time scale the shifted signal

$$x(t) \xrightarrow{\text{time shift by } t_0} x(t - t_0) \xrightarrow{\text{time scale by } \alpha} x(\alpha t - t_0)$$

2. time scale, then time shift

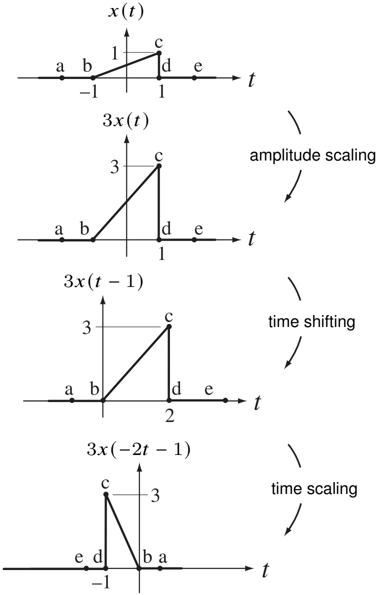
$$x(t) \xrightarrow{\text{time scale by } \alpha} x(\alpha t) \xrightarrow{\text{time shift by } t_0/\alpha} x(\alpha t - t_0)$$

Other form

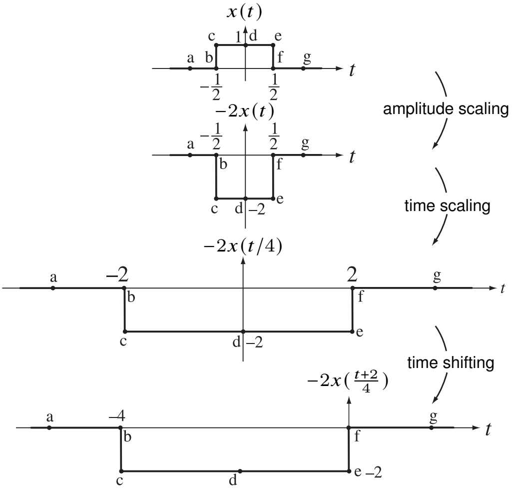
$$x\left(\frac{t - t_0}{\alpha}\right)$$

$$x(t) \xrightarrow{\text{time scale by } 1/\alpha} x(t/\alpha) \xrightarrow{\text{time shift by } t_0} x\left(\frac{t - t_0}{\alpha}\right)$$

Example: find $3x(-2t-1)$ from $x(t)$

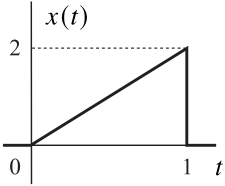


Example: find $-2x(\frac{t+2}{4})$ from $x(t)$



Exercises

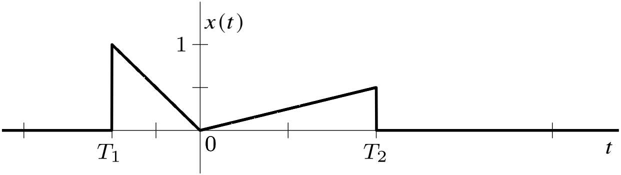
- consider the signal $x(t)$



  (a) sketch $x(t-2)$ and show it can be described mathematically as $x_d(t) = 2(t-2)$ for $2 \le t \le 3$, and equal to 0 otherwise

  (b) sketch $x(t+1)$ and show that it can be described as $x_a(t) = 2(t+1)$ for $-1 \le t \le 0$, and 0 otherwise

- sketch $y(t) = x(-3t-4)$ for the signal $x(t)$ shown with $T_1 = 2$ and $T_2 = 4$



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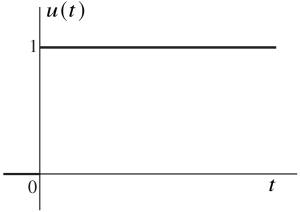
Unit step

unit step function:

$$u(t) = \begin{cases} 1 & t > 0 \\ 0, & t \leq 0 \end{cases}$$



- $u(t)$ is sometimes defined as

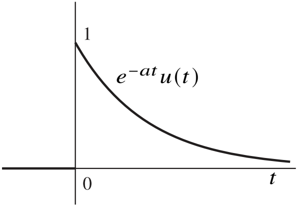
$$u(t) = \begin{cases} 1 & t > 0 \\ 0.5 & t = 0 \\ 0, & t < 0 \end{cases}$$

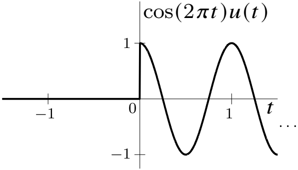
  which is convenient from a theoretical signals and systems perspective

- for real-world signals applications however, it makes no practical difference

unit-step is useful to describe causal signals

- $e^{-at}u(t)$ is zero for $t < 0$ and e^{-at} for $t \geq 0$; similarly for $\cos(2\pi t)u(t)$

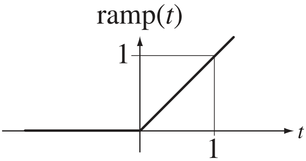




- *Unit ramp:*

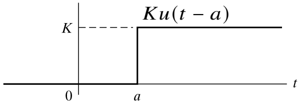
$$\text{ramp}(t) = \begin{cases} t & t > 0 \\ 0, & t \leq 0 \end{cases}$$

$$= \int_{-\infty}^{t} u(\tau)d\tau = tu(t)$$



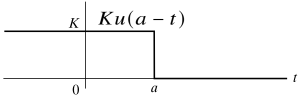
Shifted step: a step function equal to K that occurs at $t = a$ is expressed as

$$Ku(t-a) = \begin{cases} 0, & t < a \\ K, & t > a \end{cases}$$



Shifted and reversed step: a step equal to K for $t < a$ is written as

$$Ku(a-t) = \begin{cases} K, & t < a \\ 0, & t > a \end{cases}$$

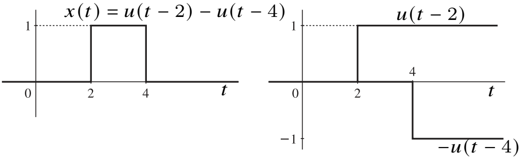


Rectangular pulse

a rectangular pulse from t_1 to t_2 can be represented as $u(t-t_1) - u(t-t_2)$

Examples:

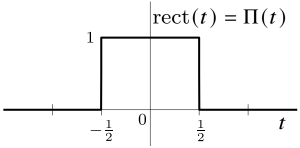
- rectangular pulse from 2 to 4



(a) (b)

- the *unit rectangle* (unit gate) is defined as

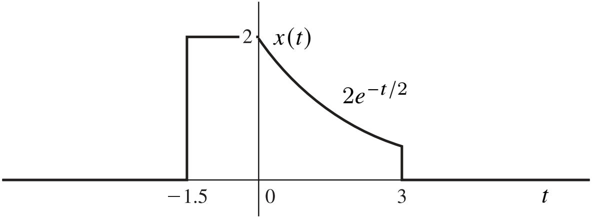
$$\mathrm{rect}(t) = \Pi(t) = u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2})$$
$$= \begin{cases} 1 & |t| < \frac{1}{2} \\ 0, & |t| \geq \frac{1}{2} \end{cases}$$



Piecewise functions

unit step functions are useful to describe piecewise functions

Example:

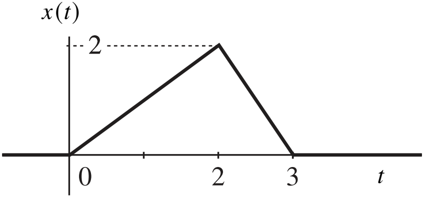


we can describe the signal $x(t)$ by a single expression valid for all t:

$$x(t) = \underbrace{2[u(t+1.5) - u(t)]}_{\text{constant part}} + \underbrace{2e^{-t/2}[u(t) - u(t-3)]}_{\text{exponential part}}$$

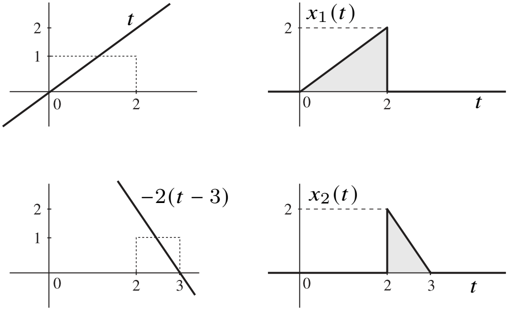
$$= 2u(t+1.5) - 2(1-e^{-t/2})u(t) - 2e^{-t/2}u(t-3)$$

Example 1.5



describe the signal $x(t)$ using the unit step function

Solution:



using line equation $x = mt + b$ and unit step functions, the signal can represented as an addition of two components:

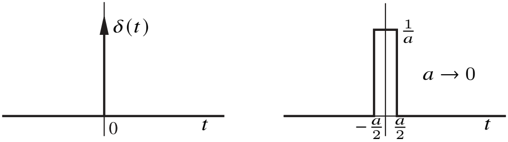
$$x_1(t) = t[u(t) - u(t-2)], \qquad x_2(t) = -2(t-3)[u(t-2) - u(t-3)]$$

therefore,

$$x(t) = x_1(t) + x_2(t) = tu(t) - 3(t-2)u(t-2) + 2(t-3)u(t-3)$$

Unit impulse

a (Dirac's) *delta function* $\delta(t)$ or unit *impulse* is an idealization of a signal that has unit area, very large near $t = 0$, and very small otherwise



- other forms of approximation can be used such as triangular; the shape is not important but the area is important
- $\delta(t)$ satisfies the property:

$$\delta(t) = 0, \quad t \neq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

undefined at $t = 0$ (not mathematically rigorous)

Properties of the impulse function

Product with impulse: for any function $g(t)$ continuous at t_0, we have

$$g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0)$$

Sampling (sifting) property: a unit impulse satisfies

$$\int_{t_1}^{t_2} g(t)\delta(t-t_0)dt = g(t_0) \qquad t_1 < t_0 < t_2$$

here, the impulse is defined as a *generalized function* (distribution), which is a function defined by its effect on other functions

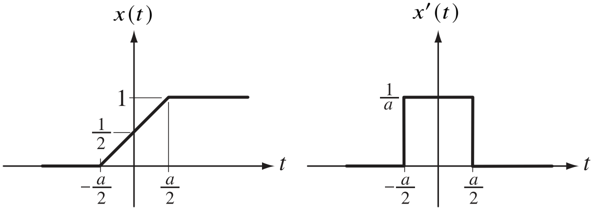
Scaling property

$$\delta\big(a(t-t_0)\big) = \frac{1}{|a|}\delta(t-t_0)$$

Unit impulse and step relation

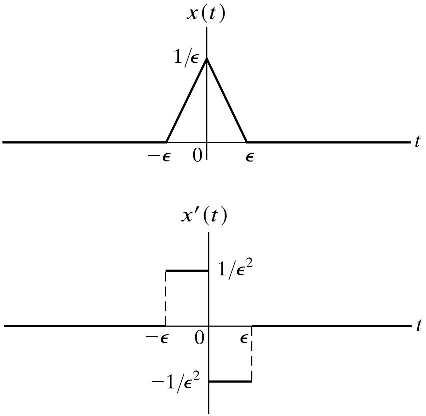
$$\frac{d}{dt}u(t - t_0) = \delta(t - t_0) \quad \text{and} \quad u(t - t_0) = \int_{-\infty}^{t} \delta(\tau - t_0)d\tau$$

Intuition



- $x(t) \to u(t)$ and $x'(t) \to \delta(t)$ as $a \to 0$
- $\delta(t)$ ($x'(t)$ as $a \to 0$) is called the *generalized derivative* of $u(t)$

The first derivative of the impulse function



(a) $x(t)$ is an impulse-generating function ($x(t) \to \delta(t)$ as $\epsilon \to 0$)

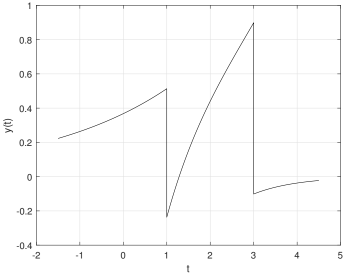
(b) $x'(t)$ shows the derivative of this impulse-generating function, which is defined as the derivative of the impulse $\delta'(t)$ as $\epsilon \to 0$; ($\delta'(t)$ is referred to as a moment function, or unit doublet)

Matlab example

the following Matlab code plots $y(t) = x(\frac{-t+3}{3}) - (3/4)x(t-1)$ over $-1.5 \leq t \leq 4.5$ where $x(t) = e^{-t}u(t)$

Matlab code

```
u = @(t) 1.0*(t>0);
x = @(t) exp(-t).*u(t); y = @(t) x((-t+3)/3)-3/4*x(t-1);
t = (-1.5:.0001:4.5); plot(t,y(t),'k');
xlabel('t'); ylabel('y(t)'); grid on;
```



Exercises

- show that

  (a) $(t^3 + 3)\delta(t) = 3\delta(t)$

  (b) $[\sin(t^2 - \pi/2)]\delta(t) = -\delta(t)$

  (c) $e^{-2t}\delta(t) = \delta(t)$

  (d) $\dfrac{\omega^2 + 1}{\omega^2 + 9}\delta(\omega - 1) = \dfrac{1}{5}\delta(\omega - 1)$

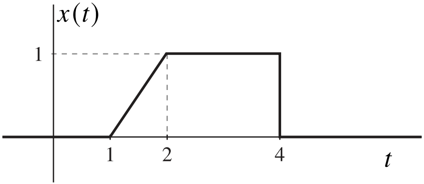
- show that

  (a) $\int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1$

  (b) $\int_{-\infty}^{\infty} \delta(t-2)\cos(\frac{\pi t}{4})dt = 0$

  (c) $\int_{-\infty}^{\infty} e^{-2(c-t)}\delta(2-t)dt = e^{-2(c-2)}$

- show that the signal



can be described as

$$x(t) = (t-1)u(t-1) - (t-2)u(t-2) - u(t-4)$$

Outline

- continuous-time signals
- signal operations
- useful CT signals
- **even and odd signals**
- signal energy and power

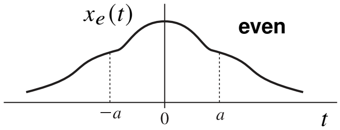
Even and odd signals

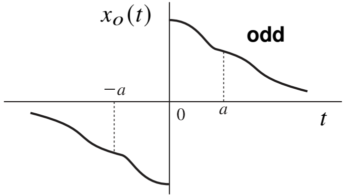
Even functions: an *even function* $x_e(t)$ is symmetrical about the vertical axis

$$x_e(t) = x_e(-t)$$

Odd functions: an *odd function* $x_o(t)$ is antisymmetrical about the vertical axis

$$x_o(t) = -x_o(-t)$$





Properties

multiplication properties

$$\begin{aligned}\text{even function} \times \text{even function} &= \text{even function}\\ \text{odd function} \times \text{odd function} &= \text{even function}\\ \text{even function} \times \text{odd function} &= \text{odd function}\end{aligned}$$

area

- for even functions

$$\int_{-a}^{a} x_e(t)dt = 2\int_{0}^{a} x_e(t)dt$$

- for odd function

$$\int_{-a}^{a} x_o(t)dt = 0$$

(under the assumption that there is no impulse at the origin)

Even and odd components

every signal $x(t)$ can expressed as

$$x(t) = \underbrace{\tfrac{1}{2}[x(t) + x(-t)]}_{\text{even part}} + \underbrace{\tfrac{1}{2}[x(t) - x(-t)]}_{\text{odd part}}$$

Examples

- the even and odd components of $e^{jt} = x_e(t) + x_o(t)$ are

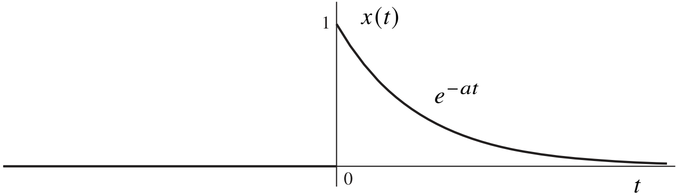
$$x_e(t) = \tfrac{1}{2}[e^{jt} + e^{-jt}] = \cos t \qquad x_o(t) = \tfrac{1}{2}[e^{jt} - e^{-jt}] = j\sin t$$

- the signal $x(t) = e^{-at}u(t)$ can be expressed as

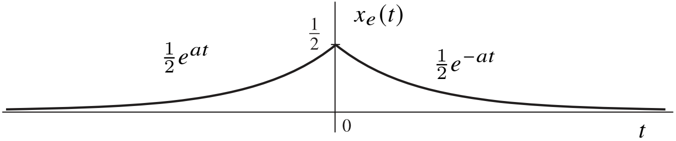
$$x(t) = x_e(t) + x_o(t)$$

where

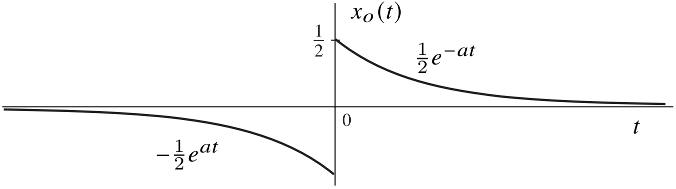
$$x_e(t) = \tfrac{1}{2}[e^{-at}u(t) + e^{at}u(-t)]$$
$$x_o(t) = \tfrac{1}{2}[e^{-at}u(t) - e^{at}u(-t)]$$


$x(t)$
1
e^{-at}
0
t

(a)


$x_e(t)$
$\frac{1}{2}$
$\frac{1}{2}e^{at}$
$\frac{1}{2}e^{-at}$
0
t

(b)


$x_o(t)$
$\frac{1}{2}$
$\frac{1}{2}e^{-at}$
$-\frac{1}{2}e^{at}$
0
t

(c)

Complex signal decomposition

Conjugate-symmetric: a signal $x(t)$ is *conjugate-symmetric* or *Hermitian* if

$$x(t) = x^*(-t)$$

Conjugate-antisymmetric: a complex signal $x(t)$ is *conjugate-antisymmetric* or or *skew Hermitian* if

$$x(t) = -x^*(-t)$$

- conjugate-symmetric signals have even real part and odd imaginary part
- conjugate-antisymmetric signals have odd real part and even imaginary part

any signal $x(t)$ can be decomposed into

$$x(t) = x_{\rm cs}(t) + x_{\rm ca}(t)$$

- $x_{\rm cs}(t) = \frac{1}{2}(x(t) + x^*(-t))$ is the conjugate-symmetric part
- $x_{\rm ca}(t) = \frac{1}{2}(x(t) - x^*(-t))$ is the conjugate-antisymmetric part

Exercise

determine the conjugate-symmetric and conjugate-antisymmetric portions of the following signals:

(a) $x_a(t) = e^{jt}$

(b) $x_b(t) = je^{jt}$

(c) $x_c(t) = \sqrt{2}e^{j(t+\pi/4)}$

Outline

- continuous-time signals
- signal operations
- useful CT signals
- even and odd signals
- **signal energy and power**

Signal energy and power

Energy of a signal

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- finite if $|x(t)| \to 0$ as $|t| \to \infty$
- infinite otherwise

(average) Power of a signal

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

- P_x is the time average (mean) of $|x(t)|^2$
- $\sqrt{P_x}$ is the *rms* (root-mean-square) value of $x(t)$

Energy and power signals

an **energy signal** is a signal with finite energy

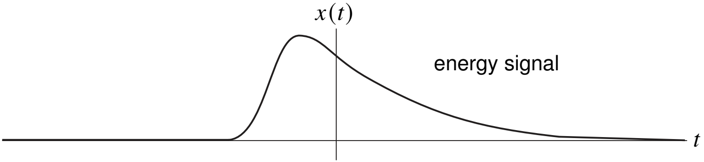
a **power signal** is a signal with finite and nonzero power

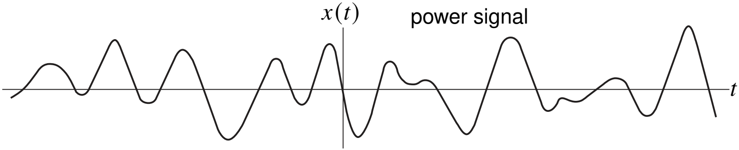
- an energy signal has zero power
- a power signal has infinite energy
- some signals are neither energy nor power signals

Power of periodic signals: a periodic signal $x(t)$ with period T_0 has power

$$P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \frac{1}{T_0} \int_{a_0}^{a_0+T_0} |x(t)|^2 dt$$

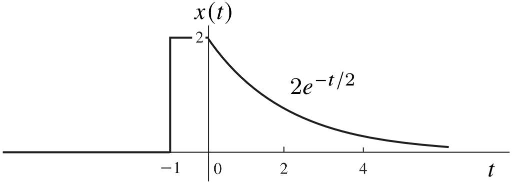
(not all power signals are periodic)


x(t)
energy signal
t

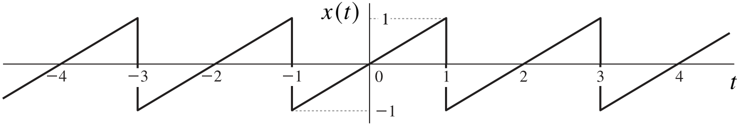

x(t)
power signal
t

Example 1.6

determine whether the signals below are energy or power signals and find their energy/power



(a)



(b)

Solution:

(a) $|x(t)|$ goes to zero as $|t| \to \infty$, hence it is an energy signal

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-1}^{0} 4dt + \int_{0}^{\infty} 4e^{-t} dt = 4 + 4 = 8$$

and $P_x = 0$

(b) $|x(t)|$ does not go to zero as $|t| \to \infty$, but it is periodic with period $T_0 = 2$, hence it is a power signal with power

$$\begin{aligned} P_x &= \frac{1}{T_0} \int_{a_0}^{a_0+T_0} |x(t)|^2 dt \\ &= \frac{1}{2} \int_{-1}^{1} |x(t)|^2 dt = \frac{1}{2} \int_{-1}^{1} t^2 dt = \frac{1}{3} \end{aligned}$$

the rms value of this signal is $1/\sqrt{3}$ and $E_x = \infty$

Example 1.7

determine the power and rms value of

(a) $x(t) = A\cos(\omega_0 t + \theta)$

(b) $x(t) = A_1\cos(\omega_1 t + \theta_1) + A_2\cos(\omega_2 t + \theta_2),\ \omega_1 \neq \omega_2$

(c) $x(t) = De^{j\omega_0 t}$

Solution:

(a) the power is

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \cos^2(\omega_0 t + \theta) dt$$
$$= \lim_{T\to\infty} \frac{A^2}{2T} \int_{-T/2}^{T/2} [1 + \cos(2\omega_0 t + 2\theta)] dt = \frac{A^2}{2} + 0 = \frac{A^2}{2}$$

- the zero term is because integral over a sinusoid is at most the area over half the cycle; thus dividing by T and letting $T \to \infty$ gives zero
- we can also integrate over the period $T_0 = 2\pi/\omega_0$:

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A^2 \cos^2(\omega_0 t + \theta) dt$$
$$= \frac{A^2}{2T_0} \int_{-T_0/2}^{T_0/2} [1 + \cos(2\omega_0 t + 2\theta)] dt = \frac{A^2}{2} + 0 = \frac{A^2}{2}$$

(second term is zero because the integration of a sinusoid over a period is zero)

- the rms value is $A/\sqrt{2}$

(b)

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2)]^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A_1^2 \cos^2(\omega_1 t + \theta_1) dt + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A_2^2 \cos^2(\omega_2 t + \theta_2) dt$$

$$+ \lim_{T\to\infty} \frac{2A_1 A_2}{T} \int_{-T/2}^{T/2} \cos(\omega_1 t + \theta_1) \cos(\omega_2 t + \theta_2) dt = \frac{A_1^2}{2} + \frac{A_2^2}{2}$$

where the third term is zero since

$$\cos(\omega_1 t + \theta_1) \cos(\omega_2 t + \theta_2)$$

$$= \cos((\omega_1 + \omega_2)t + \theta_1 + \theta_2) + \cos((\omega_1 - \omega_2)t + \theta_1 - \theta_2)$$

and the integral over a sinusoid is zero

(c)

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |D e^{j\omega_0 t}|^2 dt = \lim_{T\to\infty} \frac{|D|^2}{T} \int_{-T/2}^{T/2} dt = |D|^2$$

Power of sum of sinusoids

- the power of

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\omega_n t + \theta_n)$$

  with *distinct* frequencies and $\omega_n \neq 0$ is

$$P_x = A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2$$

- the power of

$$x(t) = \sum_{k=m}^{n} D_k e^{j\omega_k t}$$

  with *distinct* frequencies is

$$P_x = \sum_{k=m}^{n} |D_k|^2$$

Proof:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=m}^{n} \sum_{\ell=m}^{n} D_k D_\ell^* e^{(j\omega_k - \omega_\ell)t} dt$$

the integrals of the cross-product terms (when $k \neq \ell$) are finite because the integrands are periodic signals (made up of sinusoids); these terms, when divided by $T \to \infty$, yield zero; the remaining terms $(k = \ell)$ yield

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=m}^{n} |D_k|^2 \, dt = \sum_{k=m}^{n} |D_k|^2$$

Remarks

- in signal processing, when approximating a signal $x(t)$ by another signal $g(t)$, the error is defined as $e(t) = x(t) - g(t)$; the energy (or power) of $e(t)$ serves as a measure of the approximation's quality.
- in communication systems, message signals can be corrupted by noise during transmission; the quality of the received signal is assessed by the signal-to-noise power ratio
- the units of energy and power vary based on the signal type:
  - for a voltage signal $x(t)$, the energy E_x has units of volts squared-seconds ($\mathrm{V}^2\mathrm{s}$), and the power P_x has units of volts squared
  - for a current signal $x(t)$, the units are amperes squared-seconds ($\mathrm{A}^2\mathrm{s}$) for energy and amperes squared for power

Matlab example

use Matlab to approximate the energy of $x(t) = e^{-t}\cos(2\pi t)u(t)$

```
x = @(t) e^(-t).*cos(2 *pi *t).*u(t);
x_squared = @(t) x(t).*x(t);
t = (0:0.001:100);
Ex = sum(x_squared(t)*0.001)
```

[output: `Ex = 0.2567`]

a better approximation can be obtained with the `quad` function

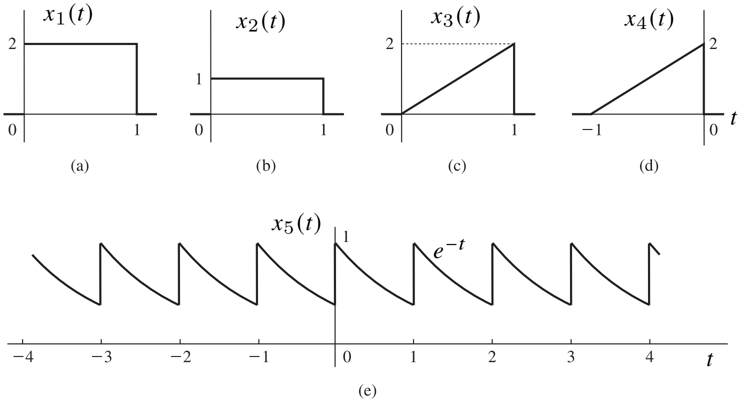
```
Ex = quad(x_squared,0,100)
```

[output: `Ex = 0.2562`]

Exercise: use Matlab to confirm that the energy of signal $y(t) = x(2t+1) + x(-t+1)$ is $E_y = 0.3768$

Exercises

- show that the energies of the signals in figure (a), (b), (c), and (d) are 4, 1, $4/3$, and $4/3$, respectively; show also that the power of the signal in (e) is 0.4323; what is the rms value of signal in figure (e)?



- find the energy of $2\,\mathrm{rect}(t/2)$
- show that the energy of $\sin(2\pi t)\,\mathrm{rect}(t/2)$ is $E_x = 1/2$
- show that an everlasting exponential e^{at} is neither an energy nor a power signal for any real value of a; however, if a is imaginary, it is a power signal with power $P_x = 1$ regardless of the value of a
- show that the power of the unit step $u(t)$ is $P_u = 1/2$
- show that if $\omega_1 = \omega_2$, then the power of

$$x(t) = A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2)$$

  is $[A_1^2 + A_2^2 + 2A_1A_2\cos(\theta_1 - \theta_2)]/2$

References

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press, chapter 1 (1.1–1.5)
- M. J. Roberts, *Signals and Systems: Analysis Using Transform Methods and MATLAB*, McGraw Hill, chapter 2