3. Time-domain analysis of continuous-time systems

- zero-input response
- the impulse response
- convolution and zero-state response
- system stability

Linear systems response

Zero-input response (ZIR)

- output $y_0(t)$ due to initial conditions alone
- input is zero

Zero-state response (ZSR)

- output due to the input $x(t)$ alone
- all initial conditions are zero

response of linear system = ZIR + ZSR

Linear time-invariant (LTI) differential system

$$\begin{aligned}\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) \\ = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{dx(t)}{dt} + b_M x(t)\end{aligned} \tag{3.1}$$

- a_i and b_i are constants; we assume $M \le N$ unless otherwise stated
- for $M > N$, the system acts as an $(M - N)$th order differentiator, which is unpractical since differentiation may greatly magnify high-frequency noise

Operator notation: D^k represents d^k/dt^k so that

$$Q(D)y(t) = P(D)x(t) \tag{3.2}$$

$$\begin{aligned}Q(D) &= D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N \\ P(D) &= b_0 D^M + b_1 D^{M-1} + \cdots + b_{M-1} D + b_M\end{aligned}$$

Zero-input solution: for the system (3.2), the ZIR is the solution to:

$$Q(D)y_0(t) = 0 \quad \text{or} \quad \left(D^N + a_1 D^{N-1} + \cdots + a_{N-1}D + a_N\right) y_0(t) = 0 \quad (3.3)$$

- a linear combination of $y_0(t)$ and its N successive derivatives is zero for all t
- possible if and only if $y_0(t)$ and all its N successive derivatives share the same form; only an exponential function $ce^{\lambda t}$ has this property
- suppose $y_0(t) = ce^{\lambda t}$ for some $c \neq 0$ and λ, then using

  $$D^j y_0(t) = \frac{d^j y_0(t)}{dt^j} = c\lambda^j e^{\lambda t}, \quad j = 1, \ldots, N$$

  in (3.3), we get

  $$c\left(\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N\right) e^{\lambda t} = cQ(\lambda) = 0$$

  hence, ce^{λ} satisfies (3.3) if $Q(\lambda) = 0$

Characteristic equation

$$Q(\lambda) = \lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N = 0 \qquad (3.4)$$

- $Q(\lambda)$ is the *characteristic polynomial* of the system
- we can express $Q(\lambda)$ in factorized form

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_N) = 0$$

- the characteristic equation has N solutions $\lambda_1, \lambda_2, \ldots, \lambda_N$ called *characteristic roots* or *characteristic values* (also *eigenvalues*) of the system
- all $c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}, \ldots, c_N e^{\lambda_N t}$ satisfy the zero-input differential equation
- the general form of the zero-input response depends on whether the roots are distinct, repeated, and/or complex

Zero-input response

Distinct roots: for distinct roots, the zero-input response of (3.3) has the form

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t}$$

- $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_N t}$ are the *characteristic modes* (also known as *natural modes* or just *modes*) of the system
- the constants c_k are constants determined by N constraints called *auxiliary conditions*; when they are given at $t = 0$, they are called *initial conditions*

Repeated roots: when the root λ_1 is repeated r times

$$Q(\lambda) = (\lambda - \lambda_1)^r (\lambda - \lambda_{r+1}) \cdots (\lambda - \lambda_N)$$

then, the solution has the form

$$y_0(t) = \left(c_1 + c_2 t + \cdots + c_r t^{r-1}\right) e^{\lambda_1 t} + c_{r+1} e^{\lambda_{r+1} t} + \cdots + c_N e^{\lambda_N t}$$

- the characteristic modes are

$$e^{\lambda_1 t}, te^{\lambda_1 t}, \ldots, t^{r-1} e^{\lambda_1 t}, e^{\lambda_{r+1} t}, \ldots, \; e^{\lambda_N t}$$

- can be generalized to multiple repeated roots

Example 3.1

find the zero-input response, $y_0(t)$, of the LTIC systems described by

(a) $(D^2 + 3D + 2)y(t) = Dx(t)$ with $y_0(0) = 0$ and $\dot{y}_0(0) = -5$

(b) $(D^2 + 6D + 9)y(t) = (3D + 5)x(t)$ with $y_0(0) = 3$ and $\dot{y}_0(0) = -7$

Solution:

(a) the characteristic equation is $Q(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ and the characteristic roots are $\lambda_1 = -1$ and $\lambda_2 = -2$ (characteristic modes are e^{-t} and e^{-2t}); therefore, the zero-input response has the form

$$y_0(t) = c_1 e^{-t} + c_2 e^{-2t}$$

taking derivative:

$$\dot{y}_0(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

to determine the constants c_1 and c_2, we use $y_0(0) = 0$ and $\dot{y}_0(0) = -5$:

$$y_0(0) = c_1 + c_2 = 0$$
$$\dot{y}_0(0) = -c_1 - 2c_2 = -5$$

solving gives $c_1 = -5$ and $c_2 = 5$; hence

$$y_0(t) = -5e^{-t} + 5e^{-2t}$$

(b) the characteristic equation is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$ and the characteristic roots are $\lambda_1 = -3, \lambda_2 = -3$ (repeated roots); the characteristic modes are e^{-3t} and te^{-3t}, and the zero-input response has the form:

$$y_0(t) = (c_1 + c_2 t)\, e^{-3t}$$

using the initial conditions $y_0(0) = 3$ and $\dot{y}_0(0) = -7$, we can show that $c_1 = 3$ and $c_2 = 2$; hence,

$$y_0(t) = (3 + 2t)e^{-3t} \quad t \geq 0$$

Complex roots forms

for a real system, complex roots appear as conjugate pairs:

$$Q(\lambda) = (\lambda - [\alpha + j\beta])(\lambda - [\alpha - j\beta]) = 0$$

complex form: for complex conjugate pair, the response is

$$y_0(t) = c_1 e^{(\alpha + j\beta)t} + c_2 e^{(\alpha - j\beta)t}$$

real-form: for a real system, c_1 and c_2 are conjugates

$$c_1 = \frac{c}{2} e^{j\theta} \quad \text{and} \quad c_2 = \frac{c}{2} e^{-j\theta}$$

so we can rewrite the response equivalently as

$$y_0(t) = c e^{\alpha t} \cos(\beta t + \theta)$$

Example 3.2

find the zero-input response, $y_0(t)$, of the LTIC system described by

$$\left(D^2 + 4D + 40\right) y(t) = (D+2)x(t)$$

with $y_0(0) = 2$ and $\dot{y}_0(0) = 16.78$

Solution: the characteristic equation is $\lambda^2 + 4\lambda + 40 = 0$; we can find the roots of a polynomial using MATLAB command:

```
r = roots([1 4 40])
```

[output: `r = -2.00+6.00i -2.00-6.00i`] hence the characteristic roots are complex $\lambda_1 = -2 + j6$ and $\lambda_2 = -2 - j6$; since $\alpha = -2$ and $\beta = 6$, the real-form solution is

$$y_0(t) = ce^{-2t}\cos(6t + \theta)$$

taking derivative, we get

$$\dot{y}_0(t) = -2ce^{-2t}\cos(6t+\theta) - 6ce^{-2t}\sin(6t+\theta)$$

to find c and θ, we use the initial conditions $y_0(0) = 2$ and $\dot{y}_0(0) = 16.78$:

$$2 = c\cos\theta$$
$$16.78 = -2c\cos\theta - 6c\sin\theta$$

solution of these two equations in two unknowns $c\cos\theta$ and $c\sin\theta$ is

$$c\cos\theta = 2 \quad \text{and} \quad c\sin\theta = -3.463$$

squaring and then adding these two equations yield

$$c^2 = (2)^2 + (-3.464)^2 = 16 \Longrightarrow c = 4$$

dividing $c\sin\theta = -3.463$ by $c\cos\theta = 2$, we have

$$\tan\theta = \frac{-3.463}{2} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{-3.463}{2}\right) = -\frac{\pi}{3}$$

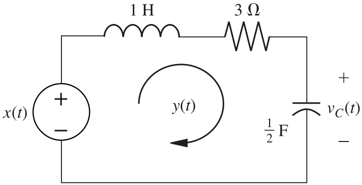
therefore,

$$y_0(t) = 4e^{-2t}\cos\left(6t - \frac{\pi}{3}\right)$$

Initial conditions

- in practice, the initial conditions are derived from the physical situation
- for example, in an RLC circuit, we may be given the conditions (initial capacitor voltages, initial inductor currents,...etc)

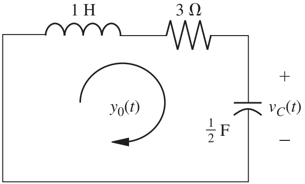
Example: find the zero-input response $y_0(t)$ (loop current) for $t \geq 0$ if $y\left(0^-\right) = 0$ and $v_C\left(0^-\right) = 5$



the differential (loop) equation relating $y(t)$ to $x(t)$ is

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

to find zero-input response $y_0(t)$, we set $x(t) = 0$ (zero-input)



we now find the initial conditions $y_0(0)$ and $\dot{y}_0(0)$; the inductor current is still zero and the capacitor voltage is still 5 volts (cannot change instantaneously); thus, $y_0(0) = 0$; to determine $\dot{y}_0(0)$, note that

$$\dot{y}_0(0) + 3y_0(0) + v_C(0) = 0$$

since $y_0(0) = 0$ and $v_C(0) = 5$, we have $\dot{y}_0(0) = -5$

the problem reduces to finding $y_0(t)$, the zero-input component of $y(t)$ of the system specified by the equation $\left(D^2 + 3D + 2\right) y(t) = 0$, with $y_0(0) = 0$ and $\dot{y}_0(0) = -5$; from example in slide 3.7, we have

$$y_0(t) = -5e^{-t} + 5e^{-2t} \quad t \geq 0$$

Meaning of 0^+ and 0^-

- conditions right before and after $t = 0$ are conditions at $t = 0^-$ and $t = 0^+$
- zero-input response $y_0(t)$ does not depend on $x(t)$, hence $y_0\left(0^-\right), \dot{y}_0\left(0^-\right), \ldots$ are identical to $y_0\left(0^+\right), \dot{y}_0\left(0^+\right), \ldots$
- in general, for the total response $y(t)$

$$y\left(0^-\right) \neq y\left(0^+\right), \dot{y}\left(0^-\right) \neq \dot{y}\left(0^+\right), \ldots$$

  because of zero-state component (*i.e.*, input affects total response at 0^+)

Example: in the previous example, we have

$$\dot{y}(0^-) + 3y(0^-) + v_C(0^-) = 0 \quad \text{(at } t = 0^- \; x(t) = 0\text{)}$$
$$\dot{y}(0^+) + 3y(0^+) + v_C(0^+) = 10 \quad \text{(at } t = 0^+ \; x(t) = 10\text{)}$$

- the loop (inductor) current and capacitor voltage cannot change instantaneously; hence, $y(0^+) = y(0^-) = 0$ and $v_C(0^+) = v_C(0^-) = 5$
- substituting these values into the above we have

$$y(0^-) = 0, \;\; y(0^+) = 0 \quad \text{and} \quad \dot{y}(0^-) = -5, \;\; \dot{y}(0^+) = 5$$

  we see that $\dot{y}(0^-) \neq \dot{y}(0^+) = 5$ for the total response

Using MATLAB to find zero-input response

using initial conditions $y_0(0) = 3$ and $\dot{y}_0(0) = -7$, apply MATLAB's `dsolve` command to determine the zero-input response to

$$\left(D^2 + 4D + k\right) y(t) = (3D + 5)x(t)$$

when: (a) $k = 3$, (b) $k = 4$, and (c) $k = 40$

(a) `yo = dsolve('D2y+4*Dy+3*y=0','y(0)=3','Dy(0)=-7','t')`

[output: `yo = 1/exp(t) + 2/exp(3*t)`]

(b) `yo = dsolve('D2y+4*Dy+4*y=0','y(0)=3','Dy(0)=-7','t')`

[output: `yo = 3/exp(2*t) - t/exp(2*t)`]

(c) `yo = dsolve('D2y+4*Dy+40*y=0','y(0)=3','Dy(0)=-7','t')`

[output: `yo = (3*cos(6*t))/exp(2*t) - sin(6*t)/(6*exp(2*t))`]

Exercises

- find the zero-input response of an LTIC system described by $(D+5)y(t) = x(t)$ if the initial condition is $y(0) = 5$
  [**Answer:** $y_0(t) = 5e^{-5t} \quad t \geq 0$]

- solve

$$\left(D^2 + 2D\right) y_0(t) = 0$$

  with $y_0(0) = 1$ and $\dot{y}_0(0) = 4$ [**Answer:** $y_0(t) = 3 - 2e^{-2t} \quad t \geq 0$]

- for the circuit in slide 3.12, the inductance $L = 0$ and the initial capacitor voltage $v_C(0) = 30$ volts; show that the zero-input component of the loop current is given by $y_0(t) = -10e^{-2t/3}$ for $t \geq 0$

Outline

- zero-input response
- **the impulse response**
- convolution and zero-state response
- system stability

Impulse response

- the (unit) **impulse response**, denoted by $h(t)$, is the output of the system when the input is $x(t) = \delta(t)$ and all initial conditions are zero
- an LTI system is causal if and only if $h(t) = 0$ for $t < 0$

Finding the impulse response for linear differential system

$$\begin{aligned}&\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) y(t) \\ &\quad = \left(b_0 D^M + b_1 D^{M-1} + \cdots + b_{M-1} D + b_M\right) x(t)\end{aligned} \tag{3.5}$$

- the impulse response satisfies

$$\begin{aligned}&\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) h(t) \\ &\quad = \left(b_0 D^M + b_1 D^{M-1} + \cdots + b_{M-1} D + b_M\right) \delta(t)\end{aligned} \tag{3.6}$$

- all zero initial conditions at $t = 0^-$ $(D^j h(0^-) = 0)$

- an impulse input $\delta(t)$ creates nonzero initial conditions (energy storages) instantaneously within the system at $t = 0^+$
- the created initial condition creates output consisting of system's characteristic modes for $t \geq 0^+$

Impulse response form

- for $M = N$, the impulse response has the form

$$h(t) = b_0\delta(t) + \text{characteristic modes terms}$$

  where b_0 is coefficient of D^N in $P(D)$

- the case $M < N$ is similar to $M = N$ with $b_0 = 0$; hence the impulse response has the form

$$h(t) = \text{characteristic modes terms}$$

- if $M > N$ we can get impulse derivatives at $t = 0$ (impractical case)

Example 3.3 (impulse matching)

find the impulse response $h(t)$ specified by

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

Solution: since $b_0 = 0$, $h(t)$ consists of the characteristic modes only; the characteristic polynomial is $\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3)$ and the roots are -2 and -3; hence, the impulse response $h(t)$ has the form:

$$h(t) = \left(c_1 e^{-2t} + c_2 e^{-3t}\right) u(t)$$

letting $x(t) = \delta(t)$ and $y(t) = h(t)$ in the differential equation, we obtain

$$\ddot{h}(t) + 5\dot{h}(t) + 6h(t) = \dot{\delta}(t) + \delta(t)$$

we have $h(0^-) = 0$ and $\dot{h}(0^-) = 0$, but the application of an impulse at $t = 0$ creates new initial conditions at $t = 0^+$; let $h\left(0^+\right) = K_1$ and $\dot{h}\left(0^+\right) = K_2$

moreover, the jump discontinuities in $h(t)$ and $\dot{h}(t)$ at $t = 0$ creates impulse terms $\dot{h}(0) = K_1\delta(t)$ and $\ddot{h}(0) = K_1\dot{\delta}(t) + K_2\delta(t)$

substituting in the impulse diff. equation and matching the coefficients of impulse terms on both sides yields

$$K_1 = 1, \quad 5K_1 + K_2 = 1 \quad \Longrightarrow \quad K_1 = 1,\ K_2 = -4$$

so $h\left(0^+\right) = K_1 = 1$ and $\dot{h}\left(0^+\right) = K_2 = -4$

using these initial conditions $h(t) = (c_1e^{-2t} + c_2e^{-3t})u(t)$, we have

$$\begin{aligned} h(0^+) &= c_1 + c_2 = 1 \\ \dot{h}(0^+) &= -2c_1 - 3c_1 = -4 \end{aligned}$$

these two simultaneous equations yield $c_1 = -1$ and $c_2 = 2$; therefore,

$$h(t) = \left(-e^{-2t} + 2e^{-3t}\right) u(t)$$

Simplified impulse matching method

for an LTIC system with $M \leq N$, the unit impulse response $h(t)$ has the form:

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]\,u(t)$$

- b_0 is coefficient of D^N in $P(D)$
- $y_n(t)$ is a linear combination of the characteristic modes of the system subject to the initial conditions:

  $$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) = \cdots = y_n^{(N-2)}(0) = 0 \quad \text{and} \quad y_n^{(N-1)}(0) = 1$$

  where $y_n^{(k)}(0)$ is the value of the kth derivative of $y_n(t)$ at $t = 0$
- for example:

$$N = 1 : y_n(0) = 1$$
$$N = 2 : y_n(0) = 0, \dot{y}_n(0) = 1$$
$$N = 3 : y_n(0) = \dot{y}_n(0) = 0, \ddot{y}_n(0) = 1, \quad \text{...etc}$$

Example 3.4

determine the unit impulse response $h(t)$ for a system specified by the equation

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

Solution: this is a second-order system ($N = 2$) having the characteristic polynomial $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$ and the characteristic roots are $\lambda = -1$ and $\lambda = -2$; thus,

$$y_n(t) = c_1 e^{-t} + c_2 e^{-2t}$$

differentiation of this equation yields $\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$; using the initial conditions $\dot{y}_n(0) = 1$ and $y_n(0) = 0$, we have

$$y_n(0) = 0 = c_1 + c_2$$
$$\dot{y}_n(0) = 1 = -c_1 - 2c_2$$

solving gives $c_1 = 1$ and $c_2 = -1$; thus, $y_n(t) = e^{-t} - e^{-2t}$

since $P(D) = D$ and $b_0 = 0$, we have

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]\, u(t) = [Dy_n(t)]\, u(t) = (-e^{-t} + 2e^{-2t})u(t)$$

Using MATLAB to find the impulse response

use MATLAB to determine the impulse response $h(t)$ for an LTIC system specified by the differential equation

$$\left(D^2 + 3D + 2\right) y(t) = Dx(t)$$

- a second-order system with $b_0 = 0$
- first we find the zero-input component for initial conditions $y\left(0^-\right) = 0$, and $\dot{y}\left(0^-\right) = 1$
- since $P(D) = D$, the zero-input response is differentiated and the impulse response immediately follows as $h(t) = 0\delta(t) + [Dy_n(t)]\, u(t)$

```
y_n = dsolve('D2y+3*Dy+2*y=0','y(0)=0','Dy(0)=1','t');
h = diff(y_n)
```

[output: `h = 2/exp(2*t) - 1/exp(t)`]

Exercises

find the impulse response of LTIC systems described by the following equations

(a) $(D+2)y(t) = (3D+5)x(t)$

(b) $D(D+2)y(t) = (D+4)x(t)$

(c) $\left(D^2+2D+1\right)y(t) = Dx(t)$

Answers:

(a) $3\delta(t) - e^{-2t}u(t)$

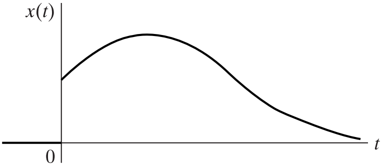
(b) $\left(2 - e^{-2t}\right)u(t)$

(c) $(1-t)e^{-t}u(t)$

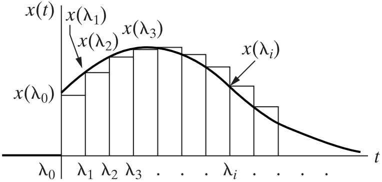
Outline

- zero-input response
- the impulse response
- **convolution and zero-state response**
- system stability

Derivation of zero-state response of LTI system



we can approximate $x(t)$ by a series of rectangular pulses of uniform width $\Delta\lambda$



$$x(t) = x_0(t) + x_1(t) + \cdots + x_i(t) + \cdots$$

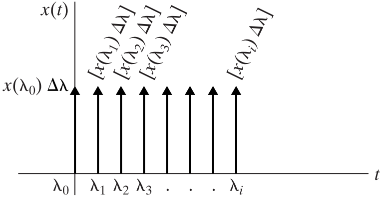
where $x_i(t)$ is a rectangular pulse that equals $x(\lambda_i)$ between λ_i and λ_{i+1} and is zero elsewhere

Observe that the ith pulse can be expressed in terms of step functions; that is,

$$x_i(t) = x(\lambda_i)\{u(t-\lambda_i) - u(t-(\lambda_i+\Delta\lambda))\}$$

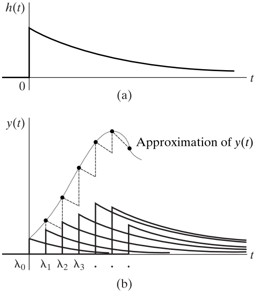
The next step in the approximation of $x(t)$ is to make $\Delta\lambda$ small enough that the ith component can be approximated by an impulse function of strength $x(\lambda_i)\Delta\lambda$

$$\lim_{\Delta\lambda\to 0} \frac{\{u(t-\lambda_i) - u(t-(\lambda_i+\Delta\lambda))\}}{\Delta\lambda} = \delta(\lambda-\lambda_i)$$



$$x(t) = x(\lambda_0)\Delta\lambda\delta(t-\lambda_0) + x(\lambda_1)\Delta\lambda\delta(t-\lambda_1) + \cdots$$
$$+x(\lambda_i)\Delta\lambda\delta(t-\lambda_i) + \cdots$$

using linearity and time-invariance the response function $y(t)$ consists of the sum of a series of uniformly delayed impulse responses



$$
\begin{aligned}
y(t) &= x(\lambda_0)\,\Delta\lambda h(t-\lambda_0) + x(\lambda_1)\,\Delta\lambda h(t-\lambda_1) \\
&\quad + x(\lambda_2)\,\Delta\lambda h(t-\lambda_2) + \cdots + x(\lambda_i)\,\Delta\lambda h(t-\lambda_i) + \cdots \\
&= \sum_{i=0}^{\infty} x(\lambda_i)\, h(t-\lambda_i)\,\Delta\lambda
\end{aligned}
$$

As $\Delta\lambda \to 0$, the summation approaches a continuous integration, hence

$$\sum_{i=0}^{\infty} x(\lambda_i)\, h(t-\lambda_i)\, \Delta\lambda \to \int_0^{\infty} x(\lambda) h(t-\lambda) d\lambda$$

Therefore,

$$y(t) = \int_0^{\infty} x(\lambda) h(t-\lambda) d\lambda$$

If $x(t)$ exists over all time, then the lower limit becomes $-\infty$; thus, in general

$$y(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda \quad \text{(Convolution integral)}$$

the convolution integral is often is written in a shorthand notation:

$$y(t) = x(t) * h(t)$$

where the asterisk signifies the convolution operation read as "$h(t)$ is convolved with $x(t)$ " and implies that

Zero-state response of LTI systems

the zero-state response is the convolution of $x(t)$ and $h(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

if $x(t)$ and $h(t)$ are both causal, then the response $y(t)$ is also causal:

$$y(t) = x(t) * h(t) = \begin{cases} \int_{0-}^{t} x(\tau)h(t-\tau)d\tau & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Example 3.5

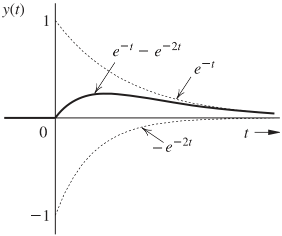
for an LTIC system with the unit impulse response $h(t) = e^{-2t}u(t)$, determine the response $y(t)$ for the input $x(t) = e^{-t}u(t)$

Solution: notice that both $x(t)$ and $h(t)$ are causal and $x(\tau) = e^{-\tau}u(\tau)$ and $h(t-\tau) = e^{-2(t-\tau)}u(t-\tau)$, therefore,

$$
\begin{aligned}
y(t) &= \int_0^t e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau \\
&= e^{-2t}\int_0^t e^{\tau}d\tau = e^{-2t}\left(e^t - 1\right) = e^{-t} - e^{-2t} \quad t \geq 0
\end{aligned}
$$

thus,

$$y(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$



Convolution properties

Convolution integral of two functions $x_1(t)$ and $x_2(t)$ is

$$x_1(t) * x_2(t) \triangleq \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$$

Commutative

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

Distributive

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

Associative

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

Shift property: if $x_1(t) * x_2(t) = y(t)$ then

$$x_1(t-t_1) * x_2(t-t_2) = y(t-t_1-t_2)$$

Convolution with an impulse

$$x(t) * A\delta(t-t_0) = Ax(t-t_0)$$

Differentiation property: if $y(t) = x_1(t) * x_2(t)$ then

$$\dot{y}(t) = \dot{x}_1(t) * x_2(t) = x_1(t) * \dot{x}_2(t)$$

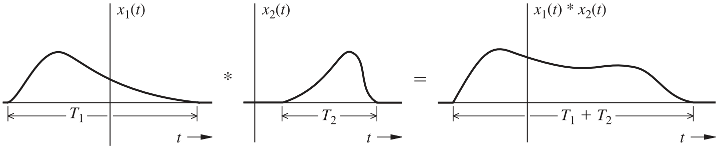
Scaling property: if $y(t) = x_1(t) * x_2(t)$ then

$$y(at) = |a|\, x_1(at) * x_2(at)$$

Area property: if $y(t) = x_1(t) * x_2(t)$ then

$$\text{area of } y = (\text{area of } x_1) \times (\text{area of } x_2)$$

Width property: if the durations (widths) of $x_1(t)$ and $x_2(t)$ are T_1 and T_2, then the duration (width) of $x_1(t) * x_2(t)$ is $T_1 + T_2$



Example 3.6 (convolution table)

the convolution of common pairs of functions are already known and can be found from the convolution table

use the convolution table to find the following convolutions:

(a) $e^{-t}u(t) * u(t)$
(b) $e^{-t}u(t) * e^{-t}u(t)$
(c) $e^{-t}u(t) * e^{-2t}u(t)$
(d) $e^{-t}u(t) * \sin(3t)u(t)$

Solution:

(a) $(1 - e^{-t})\,u(t)$
(b) $e^{-t}u(t) * e^{-t}u(t) = te^{-t}u(t)$
(c) $e^{-t}u(t) * e^{-2t}u(t) = \left(e^{-t} - e^{-2t}\right)u(t)$
(d) we use pair 12 (in Table) with $\alpha = 0, \beta = 3, \theta = -90^\circ$ and $\lambda = -1$: this gives $\phi = \tan^{-1}\left[\frac{-3}{-1}\right] = -108.4^\circ$ and

$$\begin{aligned}\sin(3t)u(t) * e^{-t}u(t) &= \frac{(\cos 18.4^\circ)e^{-t} - \cos(3t+18.4^\circ)}{\sqrt{10}}u(t) \\ &= \frac{0.9486e^{-t} - \cos(3t+18.4^\circ)}{\sqrt{10}}u(t)\end{aligned}$$

Example 3.7

an LTI system has an impulse response $h(t) = 2e^{-3t}u(t)$; determine the (zero-state) response of the system if the input is $x(t) = u(t) - u(t - 1/3)$

Solution: for input $x(t) = u(t) - u(t - 1/3)$, we have

$$\begin{aligned} y(t) = h(t) * x(t) &= h(t) * u(t) - h(t) * u(t - 1/3) \\ &= 2(e^{-3t}u(t) * u(t)) - 2(e^{-3t}u(t) * u(t - 1/3)) \end{aligned}$$

using $e^{\lambda_1 t}u(t) * e^{\lambda_2 t}u(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}u(t)$ $(\lambda_1 \neq \lambda_2)$ and the shift property $x_1(t) * x_2(t - 1/3) = y(t - 1/3)$, we get

$$\begin{aligned} y(t) &= (2/3)(1 - e^{-3t})u(t) - (2/3)(1 - e^{-3(t-1/3)})u(t - 1/3) \\ &= (2/3)\left[(1 - e^{-3t})u(t) - (1 - e^{-3(t-1/3)})u(t - 1/3)\right] \end{aligned}$$

Example 3.8

use the convolution table and the differentiation property to find the zero-state response $y(t)$ of an LTI system with impulse response $h(t) = \mathrm{rect}(t)$ and input $x(t) = \mathrm{rect}(t)$ where $\mathrm{rect}(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$

Solution: from diff. property, we have

$$\begin{aligned} y''(t) = x'(t) * h'(t) &= [\delta(t+\tfrac{1}{2}) - \delta(t-\tfrac{1}{2})] * [\delta(t+\tfrac{1}{2}) - \delta(t-\tfrac{1}{2})] \\ &= \delta(t+1) - 2\delta(t) + \delta(t-1) \end{aligned}$$

integrating twice, we get

$$\begin{aligned} y'(t) &= u(t+1) - 2u(t) + u(t-1) \\ y(t) &= (t+1)u(t+1) - 2tu(t) + (t-1)u(t-1) \end{aligned}$$

or alternatively,

$$\begin{aligned} y(t) &= x'(t) * \int_{-\infty}^{t} h(\tau)d\tau \\ &= [\delta(t+\tfrac{1}{2}) - \delta(t-\tfrac{1}{2})] * \left((t+\tfrac{1}{2})[u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2})] + u(t-\tfrac{1}{2})\right) \\ &= (t+1)u(t+1) - 2tu(t) + (t-1)u(t-1) \end{aligned}$$

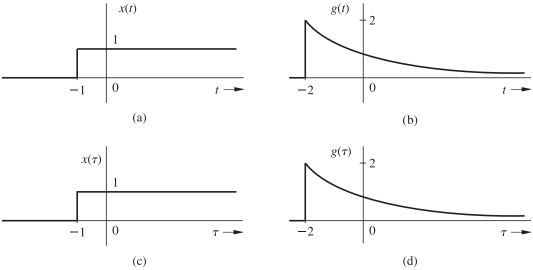
Convolution via graphical procedure

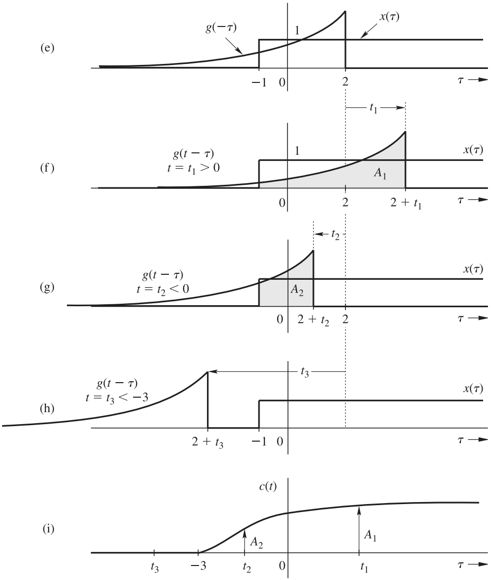
let $c(t)$ be the convolution of $x(t)$ with $g(t)$:

$$c(t) = \int_{-\infty}^{\infty} x(\tau)g(t-\tau)d\tau$$

- integration is performed with respect to τ so that t is treated as constant
- if we know graphs of $x(t)$ and $g(t)$, then we can determine $c(t)$ graphically

Illustration





$c(t_1)$ is the area A_1 and $c(t_2)$ is area A_2

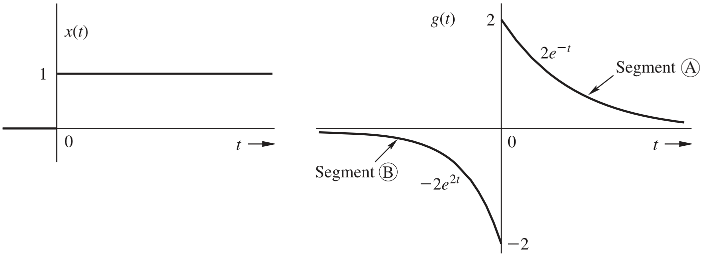
Summary of the graphical procedure

1. keep the function $x(\tau)$ fixed
2. rotate (or invert) $g(\tau)$ about the vertical axis $(\tau = 0)$ to obtain $g(-\tau)$
3. shift $g(\tau)$ along the τ axis by t_0 seconds; the shifted frame now represents $g\left(t_0 - \tau\right)$
4. the area under the product of $x(\tau)$ and $g\left(t_0 - \tau\right)$ (the shifted frame) is $c\left(t_0\right)$
5. repeat this procedure, shifting the frame by different values (positive and negative) to obtain $c(t)$ for all values of t

(if the mathematical description of $g(t)$ is simpler than that of $x(t)$, then $x(t) * g(t)$ will be easier to compute than $g(t) * x(t)$)

Example 3.9

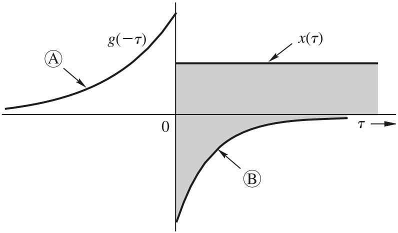
find $c(t) = x(t) * g(t)$ for the signals shown below



Solution: we have $x(t) = 1$ so that $x(\tau) = 1$; notice that

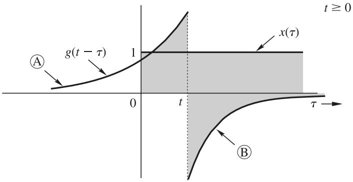
$$g(t) = \begin{cases} 2e^{-t} & \text{segment A} \\ -2e^{2t} & \text{segment B} \end{cases}$$

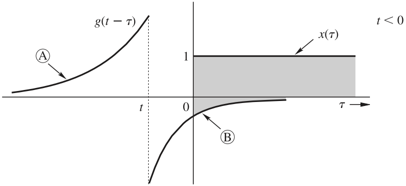
plotting $x(\tau)$ and $g(-\tau)$



we have

$$g(t-\tau) = \begin{cases} 2e^{-(t-\tau)} & \text{segment A} \\ -2e^{2(t-\tau)} & \text{segment B} \end{cases}$$





for $t \geq 0$:

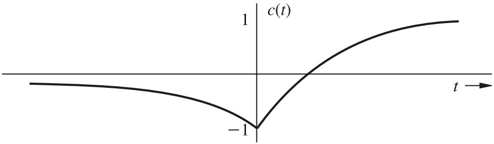
$$c(t) = \int_0^\infty x(\tau)g(t-\tau)d\tau = \int_0^t 2e^{-(t-\tau)}d\tau + \int_t^\infty -2e^{2(t-\tau)}d\tau$$
$$= 2\left(1-e^{-t}\right) - 1 = 1 - 2e^{-t}$$

for $t \leq 0$:

$$c(t) = \int_0^\infty x(\tau)g(t-\tau)d\tau = \int_0^\infty -2e^{2(t-\tau)}d\tau = -e^{2t}$$

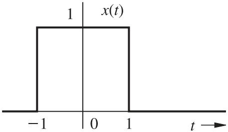
therefore,

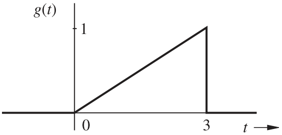
$$c(t) = \begin{cases} 1-2e^{-t} & t \geq 0 \\ -e^{2t} & t \leq 0 \end{cases}$$



Example 3.10

find $x(t) * g(t)$ for the functions $x(t)$ and $g(t)$ shown below



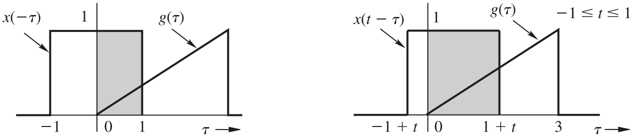


Solution: the signal $x(t)$ has a simpler mathematical description than $g(t)$; hence, we shall determine $g(t) * x(t)$:

$$c(t) = g(t) * x(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$

nonzero segments of $x(t)$ and $g(t)$ are $x(t) = 1$ and $g(t) = \frac{1}{3}t$; hence

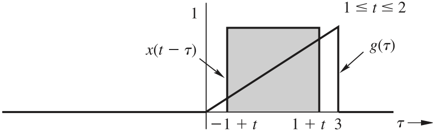
$$x(t-\tau) = 1 \quad \text{and} \quad g(\tau) = \tfrac{1}{3}\tau$$



for $-1 \le t \le 1$ the two functions overlap over the interval $(0, 1+t)$ (shaded interval) so that

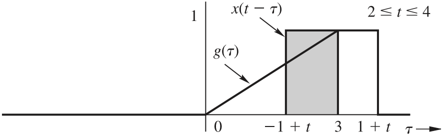
$$c(t) = \int_0^{1+t} g(\tau)x(t-\tau)d\tau = \int_0^{1+t} \tfrac{1}{3}\tau d\tau = \tfrac{1}{6}(t+1)^2, \quad -1 \le t \le 1$$

$1 \le t \le 2$:



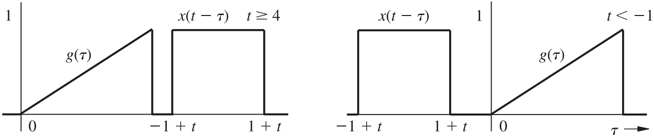
$$c(t) = \int_{-1+t}^{1+t} \frac{1}{3}\tau d\tau = \frac{2}{3}t \quad 1 \le t \le 2$$

$2 \le t \le 4$:



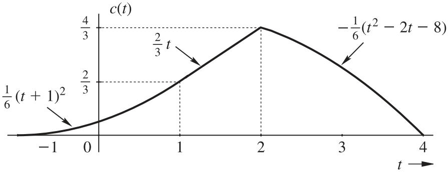
$$c(t) = \int_{-1+t}^{3} \frac{1}{3}\tau d\tau = -\frac{1}{6}\left(t^2 - 2t - 8\right) \quad 2 \le t \le 4$$

$t \geq 4$ and $t < -1$, we have $c(t) = 0$



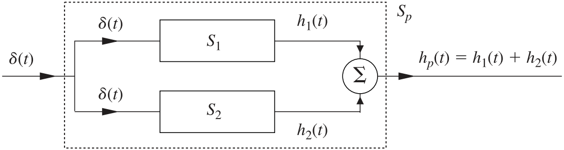
combining our results:

$$c(t) = \begin{cases} \frac{1}{6}(t+1)^2 & -1 \leq t < 1 \\ \frac{2}{3}t & 1 \leq t < 2 \\ -\frac{1}{6}\left(t^2 - 2t - 8\right) & 2 \leq t < 4 \\ 0 & \text{otherwise} \end{cases}$$

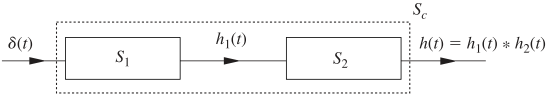


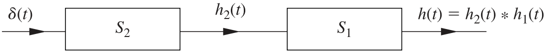
Parallel and cascade systems impulse response

Parallel connection



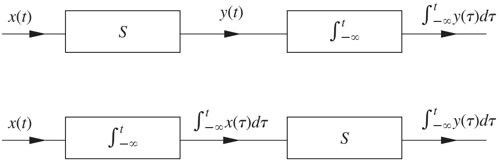
Cascade connection





Cascade systems properties

- using the commutative property of convolution, we have



- this means that $x(t) \Longrightarrow y(t)$, then $\int_{-\infty}^{t} x(\tau)d\tau \Longrightarrow \int_{-\infty}^{t} y(\tau)d\tau$
- replacing the integrator with a differentiator, we can show that $x(t) \Longrightarrow y(t)$, then $\frac{dx(t)}{dt} \Longrightarrow \frac{dy(t)}{dt}$
- the cascade system $h(t)$ with its **inverse system** $h_i(t)$ is an identity system:

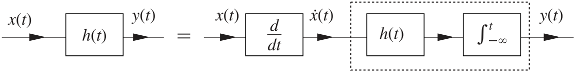
$$h(t) * h_i(t) = \delta(t)$$

Unit-step response

Unit step response: the *unit step response* (output due to step input $u(t)$) of an LTIC system with impulse $h(t)$ is

$$g(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

- in a cascade of several LTIC subsystems, changing the order of the subsystems in any manner does not affect the total impulse response



- impulse response of the dotted box is $g(t)$; thus

$$y(t) = x(t) * h(t) = \dot{x}(t) * g(t)$$

therefore, an LTIC response can also be obtained as a convolution of $\dot{x}(t)$ (the derivative of the input) with the unit step response of the system

LTI output due to exponential input

an LTIC system response $y(t)$ to an everlasting exponential e^{st} is

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = H(s) e^{st}$$

where $H(s)$ is the **transfer function** of the system:

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

- input e^{st} gives output $H(s)e^{st}$ of same form; such an input is called the *eigenfunction* (or characteristic function) of the system
- an alternate definition of the transfer function $H(s)$ of an LTIC system is

$$H(s) = \left. \frac{\text{output signal}}{\text{input signal}} \right|_{\text{input = everlasting exponential } e^{st}}$$

Practical significance

- transfer function is defined for, and is meaningful to, LTIC systems only
- it is known that every signal that can be generated in practice can be expressed as a sum of everlasting exponentials (or sinusoids)
- for example, a periodic signal $x(t)$ can be expressed as a sum of exponentials as

$$x(t) = \sum_k X_k e^{s_k t}$$

- the response $y(t)$ of an LTIC system with transfer function $H(s)$ to this input $x(t)$ is

$$y(t) = \sum_k H(s_k) X_k e^{s_k t}$$

Transfer function of LTI differential system

for a system specified by (3.2), the transfer function is given by

$$H(s) = \frac{P(s)}{Q(s)}$$

Derivation:

- to see this, we let $x(t) = e^{st}$ into (3.2) and use $y(t) = H(s)e^{st}$:

$$H(s)\left[Q(D)e^{st}\right] = P(D)e^{st}$$

- since, $D^r e^{st} = \frac{d^r e^{st}}{dt^r} = s^r e^{st}$ we have

$$P(D)e^{st} = P(s)e^{st} \quad \text{and} \quad Q(D)e^{st} = Q(s)e^{st}$$

  consequently, $H(s) = P(s)/Q(s)$

Example 3.11 (total response of LTI systems)

$$\text{LTI system total response} = \text{ZIR} + \overbrace{x(t) * h(t)}^{\text{ZSR}}$$

find the total response for the system

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

with the input $x(t) = 10e^{-3t}u(t)$ and the initial conditions $y(0^-) = 0$ and $\dot{y}(0^-) = -5$

Solution: the zero-input and the impulse response were found in slides 3.7 and 3.22 to be

$$\begin{aligned} y_0(t) &= (-5e^{-t} + 5e^{-2t}) \\ h(t) &= (2e^{-2t} - e^{-t})u(t) \end{aligned}$$

we now use the convolution table to compute the zero-state response:

$$y_{\text{zsr}}(t) = x(t) * h(t) = 10e^{-3t}u(t) * \left[2e^{-2t} - e^{-t}\right] u(t)$$

using the distributive property of the convolution, we obtain

$$\begin{aligned} y_{\text{zsr}}(t) &= 10e^{-3t}u(t) * 2e^{-2t}u(t) - 10e^{-3t}u(t) * e^{-t}u(t) \\ &= 20[e^{-3t}u(t) * e^{-2t}u(t)] - 10[e^{-3t}u(t) * e^{-t}u(t)] \end{aligned}$$

using the table (pair 4) yields

$$\begin{aligned} y(t) &= \tfrac{20}{-3-(-2)}[e^{-3t} - e^{-2t}]u(t) - \tfrac{10}{-3-(-1)}[e^{-3t} - e^{-t}]u(t) \\ &= -20(e^{-3t} - e^{-2t})u(t) + 5(e^{-3t} - e^{-t})u(t) \\ &= (-5e^{-t} + 20e^{-2t} - 15e^{-3t})u(t) \end{aligned}$$

therefore,

$$\text{total response } = \underbrace{(-5e^{-t} + 5e^{-2t})}_{\text{zero-input response}} + \underbrace{(-5e^{-t} + 20e^{-2t} - 15e^{-3t})}_{\text{zero-state response}} \quad t \geq 0$$

Natural and forced response

Natural response: the *natural response* $y_n(t)$ is the the part resulting from the combination of all the characteristic mode terms in the total response

Forced response: the *forced response* $y_\phi(t)$ is the part consisting entirely of noncharacteristic mode terms

- the forced response is the particular solution of the differential equation; it is the part of the response the form of which is determined by the input signal
- the natural response is the homogeneous solution of the differential equation, where the constants are determined such that the sum of the particular solution and the homogeneous solution satisfies the given initial condition

Example: the total response of the previous RLC example can also be expressed as

$$\text{total current} = \underbrace{\left(-10e^{-t} + 25e^{-2t}\right)}_{\text{natural response } y_n(t)} + \underbrace{\left(-15e^{-3t}\right)}_{\text{forced response } y_\phi(t)} \qquad t \geq 0$$

Exercises

- use the convolution table to show

$$e^{-2t}u(t) * \left(1 - e^{-t}\right) u(t) = (1/2)(-e^{-t} + e^{-2t})u(t)$$

- an LTIC system has the impulse response $h(t) = 6e^{-t}u(t)$; find the system response to the input:
  (a) $2u(t)$
  (b) $3e^{-3t}u(t)$
  (c) $e^{-t}u(t)$

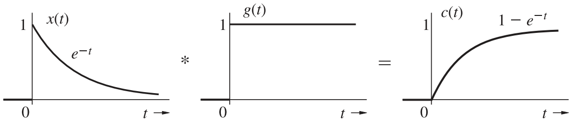
  use the convolution definition and convolution table to find the response

  Answers:
  (a) $12(1 - e^{-t})u(t)$
  (b) $9(e^{-t} - e^{-3t})u(t)$
  (c) $6te^{-t}u(t)$

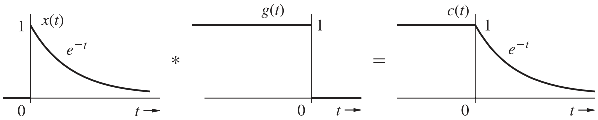
- for an LTIC system with the unit impulse response $h(t) = e^{-2t}u(t)$, determine the zero-state response $y(t)$ if the input $x(t) = \sin(3t)u(t)$
  Answer: $\frac{1}{13}[3e^{-2t} + \sqrt{13}\cos(3t - 146.32°)]u(t)$

Exercises

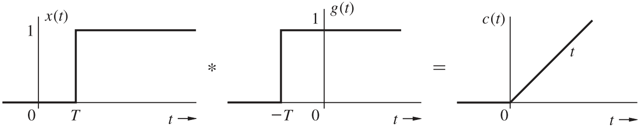
- use graphical convolution to show that $x(t) * g(t) = c(t)$



- repeat the above for



- repeat the above for



Outline

- zero-input response
- the impulse response
- convolution and zero-state response
- **system stability**

BIBO (external) stability

- system is BIBO stable if every bounded input produces a bounded output
- an LTIC system is BIBO stable if and only if $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

Examples:

- a system with $h(t) = u(t)$ is BIBO unstable since

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{0}^{\infty} d\tau = t|_0^{\infty} = \infty$$

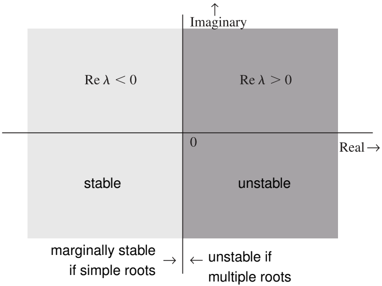
- a system with $h(t) = e^{-t}u(t)$ is BIBO stable since

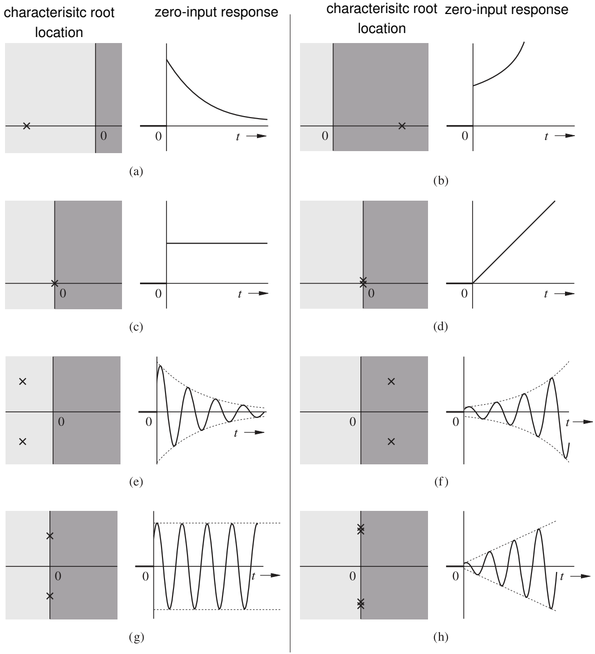
$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{0}^{\infty} e^{-t} d\tau = -e^{-t}\Big|_0^{\infty} = 1$$

Asymptotic (internal) stability

the LTIC system described by (3.1) is

1. *asymptotically stable* if, and only if, all the characteristic roots are in the LHP
2. *marginally stable* if, and only if, there are no roots in the RHP, and there are some unrepeated roots on the imaginary axis
3. *unstable* if, and only if, one or both of the following holds:
   - at least one root is in the RHP
   - there are repeated roots on the imaginary axis

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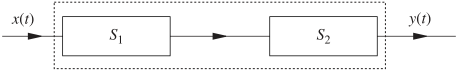
- for an LTIC system, if characteristic root λ_k is in the LHP, then the corresponding mode $e^{\lambda_k t}$ is absolutely integrable
- if λ_k is in the RHP or on the imaginary axis, then $e^{\lambda_k t}$ is not absolutely integrable

Relationship between BIBO and asymptotic stability

- an asymptotically (internally) stable system is BIBO-stable
- BIBO unstable implies asymptotically (internally) unstable
- BIBO stability does not imply asymptotic (internal) stability
- a marginally stable or asymptotically unstable *differential LTI system* is BIBO-unstable
- BIBO and asymptotic stability are equivalent if the internal and the external descriptions of a system are equivalent

Example 3.12

an LTIC system consists of two subsystems S_1 and S_2 in cascade



the impulse response of these systems are $h_1(t) = \delta(t) - 2e^{-t}u(t)$ and $h_2(t) = e^t u(t)$; determine the BIBO and asymptotic stability of the system

Solution: the composite system impulse response $h(t)$ is

$$\begin{aligned} h(t) = h_1(t) * h_2(t) &= \left[\delta(t) - 2e^{-t}u(t)\right] * e^t u(t) \\ &= e^t u(t) - 2\left[\frac{e^t - e^{-t}}{2}\right] u(t) = e^{-t}u(t) \end{aligned}$$

- composite system is BIBO-stable because $h(t)$ is absolutely integrable
- observe that the subsystem S_2 has a characteristic root 1, which lies in the RHP; hence, S_2 is asymptotically unstable;; so the whole system is unstable
- this shows that BIBO stability does not always imply asymptotic stability; however, asymptotic stability always implies BIBO stability

Example 3.13

investigate the asymptotic and the BIBO stability of the LTIC systems:

(a) $(D+1)(D^2+4D+8)y(t) = (D-3)x(t)$

(b) $(D-1)(D^2+4D+8)y(t) = (D+2)x(t)$

(c) $(D+2)(D^2+4)y(t) = (D^2+D+1)x(t)$

(d) $(D+1)(D^2+4)^2y(t) = (D^2+2D+8)x(t)$

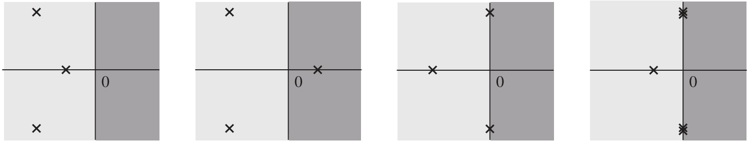
Solution: the characteristic roots of the systems are

(a) $-1, -2 \pm j2$; asymptotically stable (all roots in LHP) and BIBO-stable

(b) $1, -2 \pm j2$; unstable (one root in RHP) and BIBO-unstable

(c) $-2, \pm j2$; marginally stable (unrepeated roots on imaginary axis) and no roots in RHP; BIBO-unstable

(d) $-1, \pm j2, \pm j2$; unstable (repeated roots on imaginary axis); BIBO-unstable



(a) (b) (c) (d)

Exercises

- a system has an impulse response function shaped like a rectangular pulse, $h(t) = u(t) - u(t-1)$; is the system stable? is the system causal?
- plot the characteristic roots and determine asymptotic and BIBO stabilities for the systems:
  (a) $D(D+2)y(t) = 3x(t)$
  (b) $D^2(D+3)y(t) = (D+5)x(t)$
  (c) $(D+1)(D+2)y(t) = (2D+3)x(t)$
  (d) $(D^2+1)(D^2+9)y(t) = (D^2+2D+4)x(t)$
  (e) $(D+1)(D^2-4D+9)y(t) = (D+7)x(t)$

Answers: (a) marginally stable, but BIBO-unstable (b) unstable in both senses; (c) stable in both senses; (d) marginally stable, but BIBO-unstable; (e) unstable in both senses

References

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press, chapter 2 (2.1–2.6).
- M. J. Roberts, *Signals and Systems: Analysis Using Transform Methods and MATLAB*, McGraw Hill, chapter 5 (5.1–5.2).