8. Analysis using Laplace transform

- the transfer function
- block diagrams and stability
- frequency response
- introduction to feedback system design

Transfer function

the *transfer function* of an LTI system is the Laplace transform of the impulse response $h(t)$:

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

- the response of an LTIC system $h(t)$ to an exponential $x(t) = e^{st}$ is

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = H(s) e^{st}$$

- for LTI system with input $x(t) = e^{st}$, output is of the same form $y(t) = H(s)e^{st}$; such input is called *eigenfunction*
- an alternate definition of the transfer function $H(s)$ of an LTI system , is

$$H(s) = \left. \frac{\text{output signal}}{\text{input signal}} \right|_{\text{input}=e^{st}}$$

Zero-state response

taking Laplace transform of $y(t) = x(t) * h(t)$, we have

$$Y(s) = X(s)H(s)$$

- $H(s)$ is called transfer function because it describes in the s domain how the system "transfers" the excitation to the response
- if we know $H(s)$ and $X(s)$, then

$$y(t) = \mathcal{L}^{-1}[X(s)H(s)]$$

Transfer function of LTI differential system

$$Q(D)y(t) = P(D)x(t)$$

or

$$\begin{aligned}&\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) y(t)\\ &= \left(b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N\right) x(t)\end{aligned}$$

- the transfer function for this system is

$$H(s) = \frac{P(s)}{Q(s)}$$

- for an LTI differential system, the transfer function is simple to obtain

Example 8.1

consider an LTIC system described by the equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

find the transfer function and the zero-state response if the input $x(t) = 3e^{-5t}u(t)$

Solution: the system equation is

$$\underbrace{\left(D^2 + 5D + 6\right)}_{Q(D)} y(t) = \underbrace{(D+1)}_{P(D)} x(t)$$

therefore,

$$H(s) = \frac{P(s)}{Q(s)} = \frac{s+1}{s^2 + 5s + 6}$$

since

$$X(s) = \mathcal{L}\left[3e^{-5t}u(t)\right] = \frac{3}{s+5}$$

we have

$$Y(s) = X(s)H(s) = \frac{3(s+1)}{(s+5)\left(s^2+5s+6\right)} = \frac{-2}{s+5} - \frac{1}{s+2} + \frac{3}{s+3}$$

the inverse Laplace transform of this equation is

$$y(t) = \left(-2e^{-5t} - e^{-2t} + 3e^{-3t}\right)u(t)$$

Example 8.2

show that the transfer function of:

(a) an ideal delay of T seconds is e^{-sT}

(b) an ideal differentiator is s

(c) an ideal integrator is $1/s$

Solution:

(a) for an ideal delay of T seconds, the input $x(t)$ and output $y(t)$ are related by

$$y(t) = x(t-T) \quad \text{and} \quad Y(s) = X(s)e^{-sT}$$

therefore,

$$H(s) = \frac{Y(s)}{X(s)} = e^{-sT}$$

(b) for an ideal differentiator, the input $x(t)$ and the output $y(t)$ are related by

$$y(t) = \frac{dx(t)}{dt}$$

the Laplace transform of this equation is

$$Y(s) = sX(s) \quad \left[x\left(0^-\right) = 0 \text{ for a causal signal}\right]$$

hence

$$H(s) = \frac{Y(s)}{X(s)} = s$$

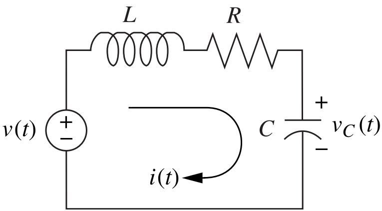
(c) for an ideal integrator with zero initial state, $y\left(0^-\right) = 0$,

$$y(t) = \int_0^t x(\tau)d\tau \quad \text{and} \quad Y(s) = \frac{1}{s}X(s)$$

therefore,

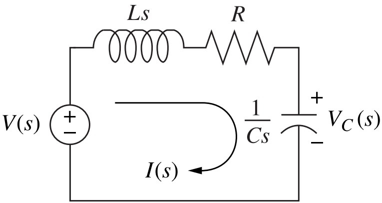
$$H(s) = \frac{1}{s}$$

Example 8.3



find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ without writing a differential equation

Solution: the Laplace circuit is



(a) **Mesh analysis:** writing a mesh equation using the impedances as we would use resistor values in a purely resistive circuit, we obtain

$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$$

solving for $I(s)/V(s)$,

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}$$

but the voltage across the capacitor, $V_C(s)$, is the product of the current and the impedance of the capacitor; thus,

$$V_C(s) = I(s)\frac{1}{Cs}$$

solving yields

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

(b) **Nodal analysis:** the currents consist of the current through the capacitor and the current flowing through the series resistor and inductor:

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0$$

solving for the transfer function, $V_C(s)/V(s)$, we arrive at the same result as before

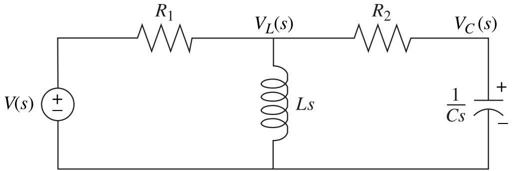
(c) **Voltage division:** the voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances; thus,

$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)} V(s)$$

solving for the transfer function, $V_C(s)/V(s)$, yields the same result as before

Example 8.4 (Complex Circuits via Nodal Analysis)

find the transfer function, $V_C(s)/V(s)$, for the circuit using nodal analysis



recall that the admittance, $Y(s)$ is the reciprocal of impedance:

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$$

when writing nodal equations, it can be more convenient to represent circuit elements by their admittance

Solution: for this problem, we sum currents at the nodes rather than sum voltages around the meshes; the sum of currents flowing from the nodes marked $V_L(s)$ and $V_C(s)$ are

$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0$$
$$CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0$$

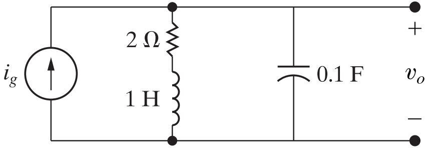
rearranging and expressing the resistances as conductances, $G_1 = 1/R_1$ and $G_2 = 1/R_2$, we obtain,

$$\left(G_1 + G_2 + \frac{1}{Ls}\right) V_L(s) \quad - G_2 V_C(s) = V(s) G_1$$
$$-G_2 V_L(s) + (G_2 + Cs)\, V_C(s) = 0$$

solving for the transfer function, $V_C(s)/V(s)$, yields

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1 G_2}{C} s}{(G_1 + G_2)\, s^2 + \frac{G_1 G_2 L + C}{LC} s + \frac{G_2}{LC}}$$

Exercise



derive the numerical expression for the transfer function V_o/I_g for the circuit shown

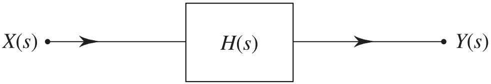
Answer: $H(s) = 10(s+2)/\left(s^2+2s+10\right)$

Outline

- the transfer function
- **block diagrams and stability**
- frequency response
- introduction to feedback system design

Block diagrams

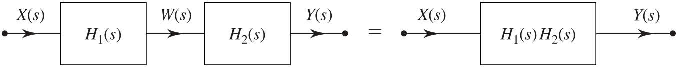
we can represent an LTI system using its transfer function using block diagrams



- $Y(s) = X(s)H(s)$
- large systems may consist of an enormous number of components or elements and conveniently represented by block diagrams

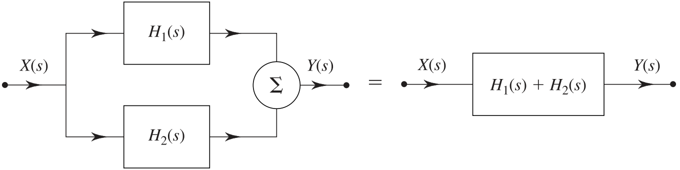
Cascade and parallel connections

Cascade interconnection



$$\frac{Y(s)}{X(s)} = \frac{W(s)}{X(s)}\frac{Y(s)}{W(s)} = H_1(s)H_2(s)$$

Parallel interconnection



Inverse systems

if $H(s)$ is the transfer function of a system $\mathcal{S}$, then the transfer function of its inverse system $\mathcal{S}_i$ is

$$H_i(s) = \frac{1}{H(s)}$$

- this follows from the fact the cascade of $\mathcal{S}$ with its inverse system $\mathcal{S}_i$ is an identity system, $h(t) * h_i(t) = \delta(t)$, implying $H(s)H_i(s) = 1$
- for example, an ideal integrator and its inverse, an ideal differentiator, have transfer functions $1/s$ and s, respectively, leading to $H(s)H_i(s) = 1$

BIBO Stability

if $H(s) = P(s)/Q(s)$ then the LTI system

- system is BIBO-stable if the poles of $H(s)$ are in LHP (excluding ω-axis)
- system is BIBO-unstable if at least one pole of $H(s)$ is *not* in LHP
- if $M > N$ ($H(s)$ is strictly improper), then the system is BIBO-unstable
  - this is because, using long division, we obtain $H(s) = R(s) + H_p(s)$, where $R(s)$ is an $(M - N)$th-order polynomial and $H_p(s)$ is a proper transfer function
  - for example,

$$H(s) = \frac{s^3 + 4s^2 + 4s + 5}{s^2 + 3s + 2} = s + \frac{s^2 + 2s + 5}{s^2 + 3s + 2}$$

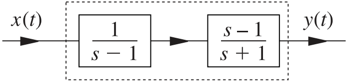
    the term s is the transfer function of an ideal differentiator
  - if we apply step function (bounded input) to this system, the output will contain an impulse (unbounded output)

Asymptotic (internal) stability

if $H(s) = P(s)/Q(s)$ and $P(s)$ and $Q(s)$ have **no common factors**, then the LTI system is

1. *asymptotically stable* if and only if all the poles of its transfer function $H(s)$ are in the LHP; the poles may be simple or repeated
2. *marginally stable* if and only if there are no poles of $H(s)$ in the RHP and some unrepeated poles on the imaginary axis
3. *unstable* if and only if either one or both of the following conditions exist:
   (i) at least one pole of $H(s)$ is in the RHP;
   (ii) there are repeated poles of $H(s)$ on the imaginary axis

Example 8.5

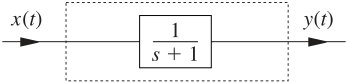


determine the BIBO and asymptotic stability of the composite (cascade) system

Solution: if the impulse responses of $\mathcal{S}_1$ and $\mathcal{S}_2$ are $h_1(t)$ and $h_2(t)$, then the impulse response of the cascade system is $h(t) = h_1(t) * h_2(t)$; hence,

$$H(s) = H_1(s)H_2(s) = \left(\frac{1}{s-1}\right)\left(\frac{s-1}{s+1}\right) = \frac{1}{s+1}$$

thus, the system is BIBO-stable since all poles are in LHP the transfer function of the system is $1/(s+1)$, without any hint of the fact that the system is housing an unstable system



to determine the asymptotic stability,

- note that $\mathcal{S}_1$ has one characteristic root at 1, and $\mathcal{S}_2$ also has one root at -1
- hence, the composite system has two characteristic roots, located at ± 1, and the system is asymptotically unstable

Exercises

- consider an LTIC system with transfer function

$$H(s) = \frac{s+5}{s^2+4s+3}$$

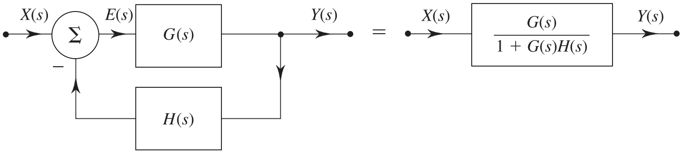
  (a) describe the differential equation relating the input $x(t)$ and output $y(t)$
  (b) find the system response $y(t)$ to the input $x(t) = e^{-2t}u(t)$ if the system is initially in zero state
  (c) determine the BIBO and asymptotic stability of this system

Answers:

(a) $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 5x(t)$
(b) $y(t) = \left(2e^{-t} - 3e^{-2t} + e^{-3t}\right)u(t)$

- show that an ideal integrator is marginally stable but BIBO-unstable

Feedback interconnection



$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Example 8.6

consider the feedback system with $G(s) = K/(s(s+8))$ and $H(s) = 1$; determine the transfer function and BIBO stability of the feedback system for each of the following cases: (a) $K = 7$; (b) $K = 16$; (c) $K = 80$

Solution: we have

$$H_{\text{feedback}}(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K/(s(s+8))}{1 + K/(s(s+8))} = \frac{K}{s^2 + 8s + K}$$

hence

(a) $H_{\text{feedback}}(s) = 7/(s^2 + 8s + 7)$, the poles are $s = -1, -7$ on LHP, hence stable

(b) $H_{\text{feedback}}(s) = 7/(s^2 + 8s + 16)$, the poles are $s = -4, -4$ on LHP, hence stable

(c) $H_{\text{feedback}}(s) = 7/(s^2 + 8s + 80)$, the poles are $s = -4 \pm 8j$ on LHP, hence stable

Matlab feedback function

consider the feedback system with $G(s) = K/(s(s+8))$ and $H(s) = 1$; use MATLAB `feedback` function to determine the transfer function for each of the following cases: (a) $K = 7$; (b) $K = 16$; (c) $K = 80$

(a)
```
>> H = tf(1,1); K = 7; G = tf([0 0 K],[1 8 0]);
TFa = feedback(G,H)
Ha =
7
-------------
s^2 + 8 s + 7
```

(b)
```
>> H = tf(1,1); K = 16; G = tf([0 0 K],[1 8 0]);
TFb = feedback(G,H)
Hb =
16
--------------
s^2 + 8 s + 16
```

(c)
```
>> H = tf(1,1); K = 80; G = tf([0 0 K],[1 8 0]);
TFc = feedback(G,H)
Hc =
80
--------------
s^2 + 8 s + 80
```

Outline

- the transfer function
- block diagrams and stability
- **frequency response**
- introduction to feedback system design

Frequency response

Frequency response: the response of an LTI system $h(t)$ to complex sinusoid $x(t) = A_x e^{j\omega t} = |A_x| e^{j(\omega t + \angle A_x)}$ is

$$
\begin{aligned}
y(t) = \int_{-\infty}^{\infty} h(\tau) A_x e^{j\omega(t-\tau)} d\tau &= H(j\omega) A_x e^{j\omega t} \\
&= |H(j\omega)||A_x| e^{j(\omega t + \angle A_x + \angle H(j\omega))}
\end{aligned}
$$

- $H(j\omega)$ is called the *frequency response* of the system
- the *amplitude* of the output is $|H(j\omega)|$ times the input amplitude
- the *phase* of the output is shifted by $\angle H(j\omega)$ with respect to the input phase
- the frequency response allows us determine the system output to any sinusoidal input using

sinusoidal input: for input $\cos(\omega t + \theta) = \mathrm{Re}(e^{j(\omega t + \theta)})$, system response is:

$$y(t) = |H(j\omega)| \cos[\omega t + \theta + \angle H(j\omega)]$$

Amplitude and phase responses

Amplitude response

- $|H(j\omega)|$ is the amplitude gain of the system called *amplitude response* or *magnitude response*
- a plot of $|H(j\omega)|$ versus ω shows the amplitude gain as a function of frequency ω

Phase response

- $\angle H(j\omega)$ is the *phase response*
- a plot of $\angle H(j\omega)$ versus ω shows how the system modifies or changes the phase of the input sinusoid

Example 8.7

an LTIC system is described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3000\frac{dy(t)}{dt} + 2 \times 10^6 y(t) = 2 \times 10^6 x(t)$$

(a) find its transfer function

(b) find $y(t)$ if $x(t) = 3e^{j\pi/2}e^{j400\pi t}$

(c) find $y(t)$ if $x(t) = 8\cos(200\pi t)$

Solution:

(a) the transfer function is

$$H(s) = \frac{2 \times 10^6}{s^2 + 3000s + 2 \times 10^6}$$

(b) the frequency response is

$$H(j\omega) = \frac{2 \times 10^6}{(j\omega)^2 + 3000(j\omega) + 2 \times 10^6} = \frac{2 \times 10^6}{2 \times 10^6 - \omega^2 + j3000\omega}$$

using $\omega = 400\pi$, we have $H(j400\pi) = 0.5272e^{-j1.46}$, hence

$$y(t) = (|H(j400\pi)| \times 3)e^{j(\angle H(j400\pi)+\pi/2)}e^{j400\pi t} = 1.582e^{j(400\pi t+0.1112)}$$

(c) we have

$$\begin{aligned} y(t) &= |H(j200\pi)| \times 8\cos(200\pi t + \angle H(j200\pi)) \\ &= 0.8078 \times 8\cos(200\pi t - 0.8654) = 6.4625\cos(200\pi t - 0.8654) \end{aligned}$$

Example 8.8

find the frequency response (amplitude and phase responses) of a system whose transfer function is

$$H(s) = \frac{s + 0.1}{s + 5}$$

also, find the system response $y(t)$ if the input $x(t)$ is

(a) $\cos 2t$

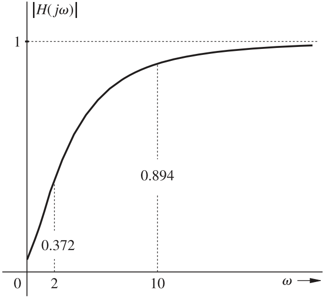
(b) $\cos(10t - 50°)$

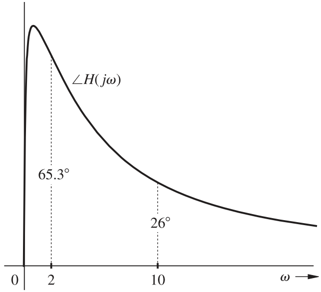
Solution:

$$H(j\omega) = \frac{j\omega + 0.1}{j\omega + 5}$$

therefore,

$$|H(j\omega)| = \frac{\sqrt{\omega^2 + 0.01}}{\sqrt{\omega^2 + 25}} \quad \text{and} \quad \angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$$



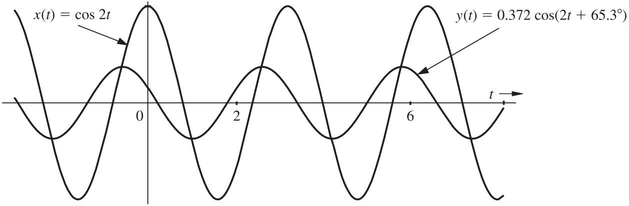


(a) for the input $x(t) = \cos 2t, \omega = 2$, and

$$|H(j2)| = \frac{\sqrt{(2)^2 + 0.01}}{\sqrt{(2)^2 + 25}} = 0.372$$
$$\angle H(j2) = \tan^{-1}\left(\frac{2}{0.1}\right) - \tan^{-1}\left(\frac{2}{5}\right) = 87.1° - 21.8° = 65.3°$$

thus, the system response to the input $\cos 2t$ is

$$y(t) = 0.372 \cos\left(2t + 65.3°\right)$$



(b) for the input $\cos(10t - 50^\circ)$, we have

$$|H(j10)| = 0.894 \quad \text{and} \quad \angle H(j10) = 26^\circ$$

therefore, the system response $y(t)$ is

$$y(t) = 0.894 \cos(10t - 50^\circ + 26^\circ) = 0.894 \cos(10t - 24^\circ)$$

the frequency response plots show that the system has highpass filtering characteristics; it responds well to sinusoids of higher frequencies (ω well above 5), and suppresses sinusoids of lower frequencies (ω well below 5)

Plotting frequency response using MATLAB

method I: use anonymous function to define the transfer function $H(s)$ and then obtain the frequency response plots by substituting $j\omega$ for s

```
>> H = @(s) (s+0.1)./(s+5); omega = 0:.01:20;
>> subplot(1,2,1); plot(omega,abs(H(1j*omega)),'k-');
>> subplot(1,2,2); plot(omega,angle(H(1j*omega))*180/pi,'k-');
```

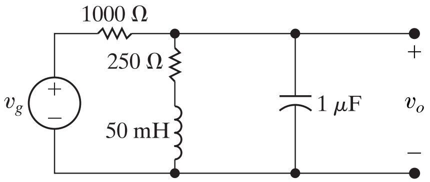
method II: we define vectors that contain the numerator and denominator coefficients of $H(s)$ and then use the `freqs` command to compute frequency response

```
>> B = [1 0.1]; A = [1 5]; omega = 0:.01:20; H = freqs(B,A,omega);
>> subplot(1,2,1); plot(omega,abs(H),'k-');
>> subplot(1,2,2); plot(omega,angle(H)*180/pi,'k-');
```

both approaches generate plots that match the previous example

Example 8.9

find the steady-state expression for v_o given that the input voltage is sinusoidal
$v_g = 120\cos(5000t + 30°)$ V



Solution: computing the transfer function using circuit analysis:

$$H(s) = \frac{V_o(s)}{V_g(s)} = \frac{1000(s+5000)}{s^2 + 6000s + 25*10^6}$$

the frequency of the voltage source is $5000\mathrm{rad/s}$ and

$$\begin{aligned} H(j5000) &= \frac{1000(5000 + j5000)}{-25*10^6 + j5000(6000) + 25 \times 10^6} \\ &= \frac{1 + j1}{j6} = \frac{1 - j1}{6} = \frac{\sqrt{2}}{6} \angle -45^\circ \end{aligned}$$

thus

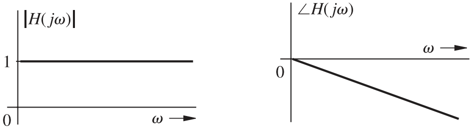
$$\begin{aligned} v_{o_{ss}} &= \frac{(120)\sqrt{2}}{6} \cos\left(5000t + 30^\circ - 45^\circ\right) \\ &= 20\sqrt{2} \cos\left(5000t - 15^\circ\right) \mathrm{V} \end{aligned}$$

(note that we can reverse the process; instead of using $H(s)$ to find $H(j\omega)$, we use $H(j\omega)$ to find $H(s)$; once we know $H(s)$, we can find the response to other excitation sources)

Ideal delay frequency response

ideal delay of T seconds $H(s) = e^{-sT}$:

$$|H(j\omega)| = 1 \quad \text{and} \quad \angle H(j\omega) = -\omega T$$

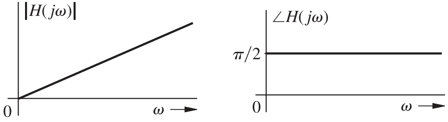


- if the input is $\cos \omega t$, the output is $\cos \omega(t - T)$
- the amplitude response (gain) is unity for all frequencies
- the phase response is linearly proportional to the frequency ω with a slope $-T$

Ideal differentiator frequency response

ideal differentiator $H(s) = s$:

$$|H(j\omega)| = \omega \quad \text{and} \quad \angle H(j\omega) = \frac{\pi}{2}$$

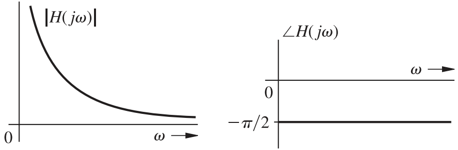


- for input $\cos \omega t$, the output is $-\omega \sin \omega t = \omega \cos[\omega t + (\pi/2)]$
- the amplitude response (gain) increases linearly with frequency ω
- the output sinusoid undergoes a phase shift $\pi/2$ with respect to the input $\cos \omega t$; thus, the phase response is constant $(\pi/2)$ with frequency
- since $|H(j\omega)| = \omega$, higher-frequency components are enhanced
- a differentiator can increase the noise is a signal, which is undesirable

Ideal integrator frequency response

an ideal integrator $H(s) = \frac{1}{s}$:

$$|H(j\omega)| = \frac{1}{\omega} \quad \text{and} \quad \angle H(j\omega) = -\frac{\pi}{2}$$



- if the input is $\cos \omega t$, the output is $(1/\omega)\sin \omega t = (1/\omega)\cos[\omega t - (\pi/2)]$
- the amplitude response is inversely proportional to ω, and the phase response is constant $(-\pi/2)$ with frequency
- because $|H(j\omega)| = 1/\omega$, the ideal integrator suppresses higher-frequency components but enhances lower-frequency components with ω
- rapidly varying noise signals are suppressed (smoothed out) by an integrator

Steady-state response to causal sinusoidal inputs

for the input $x(t) = e^{j\omega t}u(t)$, we have

$$Y(s) = X(s)H(s) = X(s)\frac{P(s)}{Q(s)} = \frac{P(s)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_N)(s-j\omega)}$$

using partial fraction expansion, we can rewrite $Y(s)$ as

$$Y(s) = \sum_{i=1}^{n} \frac{k_i}{s-\lambda_i} + \frac{H(j\omega)}{s-j\omega}$$

for some constants k_i; taking inverse Laplace transform:

$$y(t) = \underbrace{\sum_{i=1}^{n} k_i e^{\lambda_i t}u(t)}_{\text{transient component } y_{\text{tr}}(t)} + \underbrace{H(j\omega)e^{j\omega t}u(t)}_{\text{steady-state component } y_{\text{ss}}(t)}$$

- for an asymptotically stable system, the characteristic mode terms $e^{\lambda_i t}$ decay with time
- the last term $H(j\omega)e^{j\omega t}$ persists forever, and is the *steady-state* component of the response given by

$$y_{\mathrm{ss}}(t) = H(j\omega)e^{j\omega t}u(t)$$

- for a causal sinusoidal input $\cos \omega t$, the steady-state response is

$$y_{\mathrm{ss}}(t) = |H(j\omega)| \cos[\omega t + \angle H(j\omega)]u(t)$$

Exercise: find the steady-state response of an LTIC system specified by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 5x(t)$$

if the input is a causal sinusoid $20 \sin(3t + 35^\circ)\, u(t)$

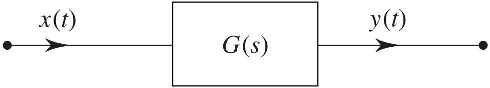
Answer: $10.23 \sin(3t - 61.91^\circ)$

Outline

- the transfer function
- block diagrams and stability
- frequency response
- **introduction to feedback system design**

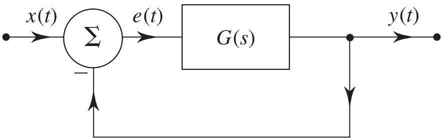
System design

systems aim to produce a specific output $y(t)$ for an input $x(t)$



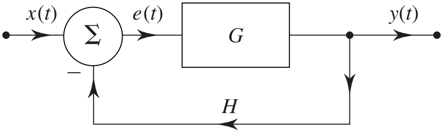
- open-loop systems should yield the desired output but may change due to aging, component replacement, or environment
- these variations can alter the output, requiring corrections at the input
- the needed input correction is the difference between actual and desired output
- feedback of the output or its function to the input may counteract variations

Feedback



- feedback systems address problems from disturbances like noise signals or environmental changes
- they aim to meet objectives within tolerances, adapting to system and environmental changes
- feedback allows supervision and self-correction against parameter variations and disturbances

Example: negative feedback amplifier



- forward amplifier gain $G = 10,000$, with $H = 0.01$ feedback, gives:

$$T = \frac{G}{1 + GH} = \frac{10,000}{1 + 100} = 99.01$$

- if G changes to $20,000$, the new gain is:

$$T = \frac{20,000}{1 + 200} = 99.5$$

- shows reduced sensitivity to forward gain G variations (changing G by 100% changes T by 0.5%)

Example: positive feedback amplifier

$$T = \frac{G}{1 - GH}$$

- for $G = 10,000$ and $H = 0.9 \times 10^{-4}$, gain T is:

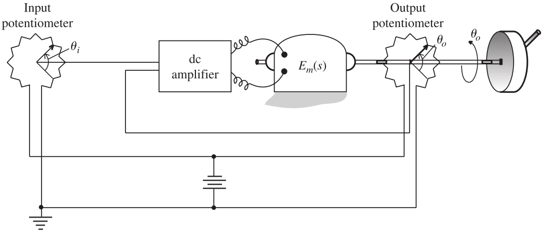
$$T = \frac{10,000}{1 - 0.9\,(10^4)\,(10^{-4})} = 100,000$$

- with $G = 11,000$, new gain is:

$$T = \frac{11,000}{1 - 0.9(11,000)\,(10^{-4})} = 1,100,000$$

- highlights sensitivity to forward gain G changes
- positive feedback increases system gain but also sensitivity to parameter changes, leading to potential instability
- for $G = 111,111$, $GH = 1$ results in $T = \infty$ and system instability

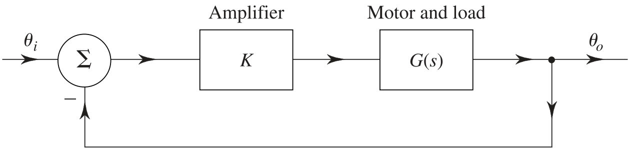
Automatic position system



the system controls the angular position of heavy objects like tracking antennas or gun mounts

- input θ_i is the desired angular position
- actual position θ_o measured by a potentiometer
- difference $\theta_i - \theta_o$ amplified and applied to motor input
- motor stops if $\theta_i - \theta_o = 0$, moves if $\theta_o \neq \theta_i$
- system controls remote object's angular position by setting input potentiometer

Block diagram of automatic position system



- amplifier gain is K (adjustable)
- motor transfer function $G(s)$ relates output angle θ_o to input voltage
- system transfer function $T(s) = \frac{KG(s)}{1+KG(s)}$
- next, we examine behavior for step and ramp inputs

Step input

- step input indicates instantaneous angle change
- we want to assess time to reach desired angle, whether it's reached smoothly or oscillates
- output $\theta_o(t)$ found for input $\theta_i(t) = u(t)$
- step input test reveals system's performance under various conditions

for step input $\theta_i(t) = u(t), \Theta_i(s) = \frac{1}{s}$,

$$\Theta_o(s) = \frac{KG(s)}{s[1 + KG(s)]}$$

assuming $G(s) = \frac{1}{s(s+8)}$, investigate system behavior for different K values

$$\Theta_o(s) = \frac{\frac{K}{s(s+8)}}{s\left[1 + \frac{K}{s(s+8)}\right]} = \frac{K}{s\left(s^2 + 8s + K\right)}$$

for $K = 7$

$$\theta_o(t) = \left(1 - \frac{7}{6}e^{-t} + \frac{1}{6}e^{-7t}\right) u(t)$$

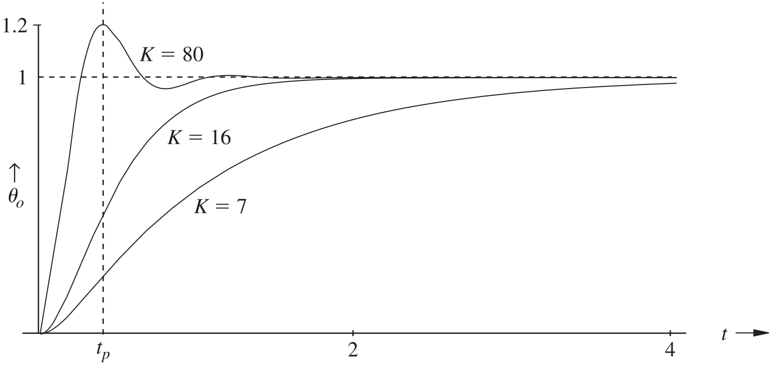
system reaches desired angle leisurely

for $K = 80$,

$$\theta_o(t) = \left[1 + \frac{\sqrt{5}}{2}e^{-4t}\cos(8t + 153^\circ)\right] u(t)$$

response for $K = 80$ reaches final position faster but with high overshoot/oscillations

- percent overshoot (PO) is 21%; peak time $t_p = 0.393$, rise time $t_r = 0.175$
- steady-state error is zero, settling time $t_s \approx 1$ second
- a good system has small overshoot, t_r, t_s, and steady-state error



to avoid oscillations in an automatic position system, choose real characteristic roots

- characteristic polynomial is $s^2 + 8s + K$
- for $K > 16$, roots are complex; for $K < 16$, roots are real
- fastest response without oscillations at $K = 16$

for $K = 16$,

$$\begin{aligned}\Theta_o(s) &= \frac{16}{s(s^2+8s+16)} = \frac{16}{s(s+4)^2} \\ &= \frac{1}{s} - \frac{1}{s+4} - \frac{4}{(s+4)^2} \\ \theta_o(t) &= \left[1 - (4t+1)e^{-4t}\right] u(t)\end{aligned}$$

system is critically damped at $K = 16$, underdamped ($K > 16$) or overdamped ($K < 16$)

Trade-off between overshoot and rise time in system response

- reducing overshoot increases rise time
- small overshoots may be acceptable for faster response
- overshoot and peak time not applicable for overdamped or critically damped cases
- adjust gain K or use compensators for stringent specifications

Ramp input

response of the system to a ramp input

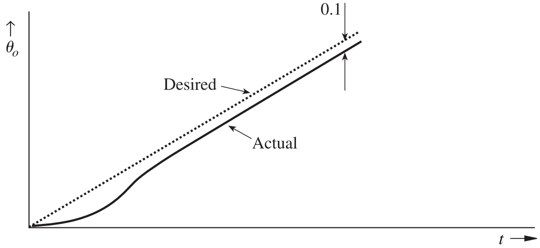
$$\theta_i(t) = tu(t)$$

when $K = 80$

$$\Theta_i(s) = \frac{1}{s^2}$$

$$\Theta_o(s) = \frac{80}{s^2(s^2+8s+80)} = -\frac{0.1}{s} + \frac{1}{s^2} + \frac{0.1(s-2)}{s^2+8s+80}$$

$$\theta_o(t) = \left[-0.1 + t + \frac{1}{8} e^{-8t} \cos(8t + 36.87°) \right] u(t)$$



response to a ramp input $\theta_i(t) = tu(t)$ with a steady-state error

- steady-state error $e_r = 0.1$ radian may be tolerable
- zero error requires compensator addition

Matlab example

using feedback system $G(s) = \frac{K}{s(s+8)}$ and $H(s) = 1$, determine step response for $K = 7, 16, 80$

- code for unit step response

```
H = tf(1,1); K = 7; G = tf([K],conv([1 0],[1 8])); Ha = feedback(G,H);
H = tf(1,1); K = 16; G = tf([K],conv([1 0],[1 8])); Hb = feedback(G,H);
H = tf(1,1); K = 80; G = tf([K],conv([1 0],[1 8])); Hc = feedback(G,H);
clf; step(Ha,'k-',Hb,'k--',Hc,'k-.');
legend('K = 7','K = 16','K = 80','Location','best');
```

- code for unit ramp response when $K = 80$

```
t = 0:.001:1.5; Hd = series(Hc,tf([1],[1 0]));
step(Hd,'k-',t); title('Unit Ramp Response');
```

Design specification

- transient specifications: overshoot, rise time, settling time for step input
- steady-state error: difference between desired and actual response in steady state
- sensitivity to system parameter variations or disturbances
- system stability under operating conditions

References

Reference:

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press, chapter 4 (4.3-4.8).

Further reading and practice exercises:

- Read section(s) 4.3, 4.5-4.6, 4.8 in the book.