11. Introduction to state-space description

- state-space modeling
- state equations from transfer function
- Laplace transform solution of state equations
- state-equations of discrete-time systems

External and internal descriptions

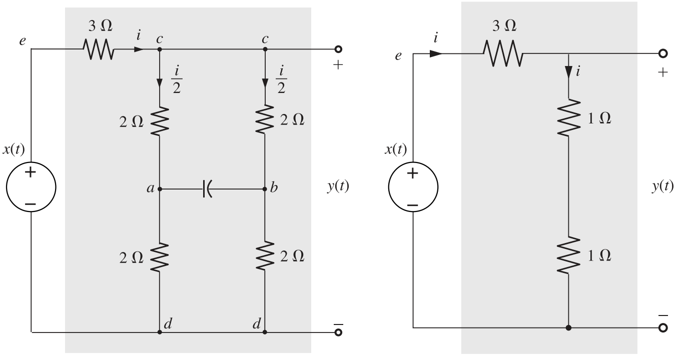
External description: a description that can be obtained from measurements at the external terminals

- the input-output description is an external description
- may not provide complete information about all signals in the systems

Internal description: an *internal description* is capable of providing complete information about all possible signals in the system

- an external description can always be found from an internal description
- the converse is not necessarily true

Example: external description



(a) (b)

output $y(t)$ will depend on input $x(t)$ and initial charge Q_0 on the capacitor

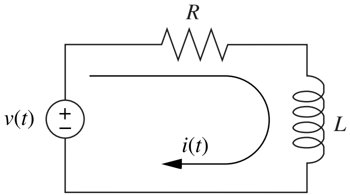
- *zero-input response:* when $x(t) = 0$ (short input)
  - currents in two 2Ω resistors at output terminals are equal and opposite
  - this is because of the balanced nature of the circuit
  - hence, $y(t) = 0$
- *zero-state response:* when $Q_0 = 0$ (short capacitor)
  - the current divides equally between branches
  - voltage across capacitor continues to remain zero and we get the equiv. circuit on right

$$y(t) = 2i(t) = \frac{2}{5}x(t)$$

- the total response $y(t) = \frac{2}{5}x(t)$
  - gives the external description
  - no external measurement can detect the presence of the capacitor

Example: state-space internal description

consider the RL network shown with an initial current of $i(0)$



- we select the current $i(t)$ as our variable
- we write the loop equation,

$$L\frac{di}{dt} + Ri = v(t)$$

- solving gives:

$$i(t) = \frac{1}{R}\left(1 - e^{-(R/L)t}\right) + i(0)e^{-(R/L)t}$$

- we can find all other variables algebraically in terms of $i(t)$ and the input voltage:

$$v_R(t) = Ri(t) \qquad \text{(output equation 1)}$$
$$v_L(t) = v(t) - Ri(t) \qquad \text{(output equation 2)}$$

  knowing the state variable, $i(t)$, and the input, $v(t)$, we can find the value, or state, of any network variable at any time, $t \geq t_0$

- these combined equations form a viable representation of the network, which we call a state-space representation

$$\frac{di}{dt} = -\frac{R}{L}i(t) + \frac{1}{L}v(t) \qquad \text{(state equation)}$$
$$v_R(t) = Ri(t) \qquad \text{(output equation 1)}$$
$$v_L(t) = v(t) - Ri(t) \qquad \text{(output equation 2)}$$

State-space description

State variables: any variables $q_1(t), \ldots, q_N(t)$ such that we can determine all signals in the system for $t \geq t_0$, given the input(s) for $t \geq t_0$ and i.c. $q_k(t_0)$

- output at t is determined completely from the states and input at t
- state variables are not unique

State-space description: the *state-space description* is an internal description where all signals in the system are expressed using state variables

- the equations relating all state variables are called *state equations*
- for input-output description, an Nth-order differential system is described by an Nth-order equation; in the state-variable approach, the same system can be described by N simultaneous first-order state equations

Linear system state-space equations

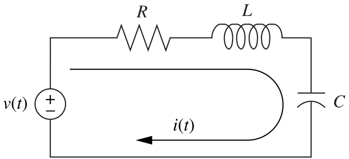
for a linear CT systems, the state and output equations can be expressed as:

$$\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{q} + \boldsymbol{B}\boldsymbol{x}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{q} + \boldsymbol{D}\boldsymbol{x}$$

- $\boldsymbol{q} = (q_1, q_2, \ldots, q_N)$ is the *state vector*
- $\boldsymbol{x} = (x_1, x_2, \ldots, x_M)$ is the *input vector*
- $\boldsymbol{A}$ is an $N \times N$ matrix
- $\boldsymbol{B}$ is an $N \times M$ matrix
- $\boldsymbol{y} = (y_1, y_2, \ldots, y_k)$ is the *output vector*
- $\boldsymbol{C}$ is an $k \times N$ matrix
- $\boldsymbol{D}$ is an $k \times M$ matrix

Example: second-order system

consider the second-order circuit



- since the network is of second order two state variables are needed; let us select the inductor current $i(t)$ and capacitor voltage $v_C(t)$
- writing the loop equation yields

$$L\frac{di}{dt} + Ri + \frac{1}{C}v_C(t) = v(t)$$

we also have

$$i(t) = C\frac{dv_C}{dt}$$

- rearranging by letting derivatives on one side and other terms on the other side:

$$\frac{di}{dt} = -(R/L)i - \frac{1}{LC}v_C(t) - (1/L)v(t)$$
$$\frac{dv_C}{dt} = (1/C)i(t)$$

- these equations are the state equations and can be solved simultaneously for the state variables, $i(t)$ and $v_C(t)$ if we know the input, $v(t)$, and the initial conditions
- from these state variables, we can solve for all other network variables, *e.g.*,

$$v_L(t) = -Ri(t) - v_c(t) + v(t)$$

this equation is an output equation

the state equations, can be written as

$$\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{q} + \boldsymbol{B}x$$

where

$$\dot{\boldsymbol{q}} = \begin{bmatrix} di/dt \\ dv_C/dt \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} -R/L & -1/LC \\ 1/C & 0 \end{bmatrix}$$

$$\boldsymbol{q} = \begin{bmatrix} i \\ v_C \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -1/L \\ 0 \end{bmatrix}; \quad x = v(t)$$

the output equation, can be written as

$$y = \boldsymbol{C}\boldsymbol{x} + Dx$$

where

$$y = v_L(t), \quad \boldsymbol{C} = \begin{bmatrix} -R & -1 \end{bmatrix}, \quad \boldsymbol{q} = \begin{bmatrix} i \\ v_C \end{bmatrix}, \quad D = 1, \quad x = v(t)$$

Non-uniqueness

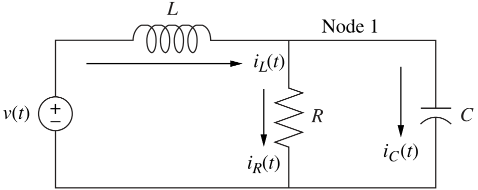
- a state-space representation is not unique
- different choices of state variables lead to a different representations

Example: if we choose $v_R(t)$ and $v_C(t)$ to be the state variables in the previous example, then state equation become:

$$\frac{dv_R}{dt} = -\frac{R}{L}v_R - \frac{R}{L}v_C + \frac{R}{L}v(t)$$
$$\frac{dv_C}{dt} = \frac{1}{RC}v_R$$

Example 11.1

find a state-space representation if the output is the current through the resistor



Solution: the following steps gives a viable representation in state-space

- select the state variables by writing the derivative equation for all energy-storage elements, that is, the inductor and the capacitor:

$$C\frac{dv_C}{dt} = i_C$$
$$L\frac{di_L}{dt} = v_L$$

  we choose the state variables as the differentiated quantities, namely v_C and i_L

- since i_C and v_L are not state variables, our next step is to express i_C and v_L as linear combinations of the state variables, v_C and i_L, and the input, $v(t)$

- using Kirchhoff's voltage and current laws at Node 1,

$$i_C = -i_R + i_L = -\frac{1}{R}v_C + i_L$$
$$v_L = -v_C + v(t)$$

  which yields i_C, v_L in terms of the state variables, v_C and i_L, and input $v(t)$

- putting things together

$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L$$
$$\frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}v(t)$$

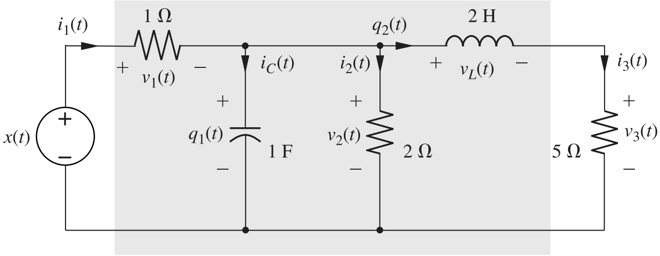
- we now find the output equation; since the output is $i_R(t)$,

$$i_R = \frac{1}{R}v_C$$

- rewritten in vector-matrix form:

$$\begin{bmatrix} \frac{dv_C}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t)$$
$$i_R = \begin{bmatrix} 1/R & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

Example 11.2



find the state equations using the state variables $q_1(t)$ (capacitor voltage) and the $q_2(t)$ (inductor current)

verify that all possible system signals at some instant t can be determined from the system state and the input at t

Solution: $\dot{q}_1$ is the current through the capacitor and $2\dot{q}_2$ is the voltage across the inductor; using KCL and KVL, we have

$$\dot{q}_1 = i_C = i_1 - i_2 - q_2 = (x - q_1) - 0.5q_1 - q_2 = -1.5q_1 - q_2 + x$$
$$2\dot{q}_2 = q_1 - v_3 = q_1 - 5q_2 \quad \Longleftrightarrow \quad \dot{q}_2 = 0.5q_1 - 2.5q_2$$

thus, the state equations are

$$\dot{q}_1 = -1.5q_1 - q_2 + x$$
$$\dot{q}_2 = 0.5q_1 - 2.5q_2$$

in matrix form, we have

$$\underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{\boldsymbol{q}}} = \underbrace{\begin{bmatrix} -1.5 & -1 \\ 0.5 & -2.5 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\boldsymbol{q}} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\boldsymbol{B}} x$$

once we solve for q_1 and q_2 at t, we can use KCL and KVL to find any possible signal (current/voltage) in the circuit at t:

$$
\begin{aligned}
i_1 &= (x - q_1)/1 & \qquad i_C &= (x - q_1)/1 - q_1/2 - q_2 \\
v_1 &= x - q_1 & i_3 &= q_2 \\
v_2 &= q_1 & v_3 &= 5q_2 \\
i_2 &= q_1/2 & v_L &= q_1 - 5q_2
\end{aligned}
$$

if the outputs are $y_1 = v_1$ and $y_2 = i_C$, then the output equations are

$$
\begin{aligned}
y_1 &= x - q_1 \\
y_2 &= -(3/2)q_1 - q_2 + x
\end{aligned}
$$

or in matrix form:

$$
\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{bmatrix} -1 & 0 \\ -3/2 & -1 \end{bmatrix}}_{\boldsymbol{C}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\boldsymbol{q}} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\boldsymbol{D}} x
$$

Mesh-current method procedure

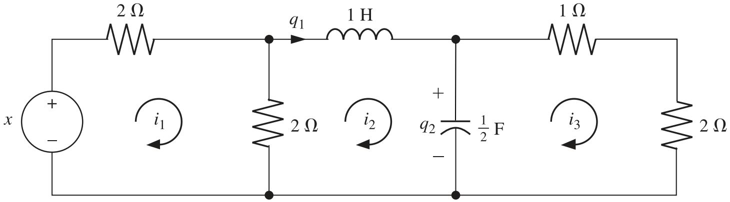
1. choose all capacitor voltages and inductor currents to be the state variables

2. write the mesh-loop currents equations and express the state variables and their first derivatives in terms of the loop currents

3. eliminate all variables other than state variables (and their first derivatives)

Example 11.3

write the state equations for the circuit shown



Solution: there is one inductor and one capacitor in the network; thus, we choose the inductor current q_1 and the capacitor voltage q_2 as the state variables

loop equations:

$$4i_1 - 2i_2 = x$$
$$2(i_2 - i_1) + \dot{q}_1 + q_2 = 0$$
$$-q_2 + 3i_3 = 0$$

loop currents and state variables relation:

$$q_1 = i_2$$
$$\frac{1}{2}\dot{q}_2 = i_2 - i_3$$

from the second loop equation, we have

$$\dot{q}_1 = 2(i_1 - i_2) - q_2 = -i_2 + 2i_1 - i_2 - q_2$$

using $q_1 = i_2$ and $2i_1 - i_2 = 1/2x$, we can eliminate i_1 and i_2, to obtain

$$\dot{q}_1 = -q_1 - q_2 + \frac{1}{2}x$$

using $q_1 = i_2$ and $\frac{1}{2}\dot{q}_2 = i_2 - i_3$ and the last loop equation, we get

$$\dot{q}_2 = 2q_1 - \frac{2}{3}q_2$$

hence the state equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} x$$

Why state-space?

- suited for (large) multiple-input, multiple-output (MIMO) systems
  - such as a vehicle with input direction and input velocity yielding an output direction and an output velocity
- compact matrix notation along with powerful techniques of linear algebra greatly facilitates complex manipulations
  - without such features, many important results of modern system theory would have been difficult to obtain
- provides useful information about a system even if not solved explicitly
- used to simulate complex systems of high order with multiple inputs/outputs

Outline

- state-space modeling
- **state equations from transfer function**
- Laplace transform solution of state equations
- state-equations of discrete-time systems

Differential equation to transfer function

$$
\begin{aligned}
&(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) \\
&= (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N) x(t)
\end{aligned}
$$

recall the transfer function for this system is

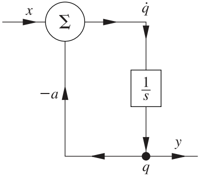
$$
H(s) = \frac{b_0 s^N + b_1 s^{N-1} + \cdots + b_{N-1} s + b_N}{s^N + a_1 s^{N-1} + \cdots + a_{N-1} s + a_N}
$$

State equations from transfer function

- state equations of LTI systems can be determined from the transfer function
- integrator output q serves as a natural state variable and integrator input is $\dot{q}$

Illustration:

$$H(s) = \frac{1}{s+a}$$



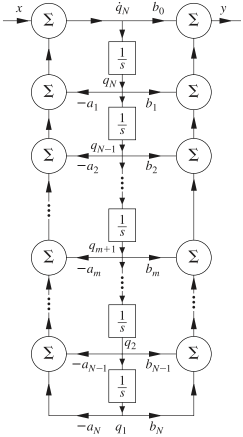
$$\dot{q} = -aq + x \quad \text{and} \quad y = q$$

(different realizations lead to different state-space descriptions of the same system)

Direct form II state equations

$$H(s) = \frac{b_0 s^N + b_1 s^{N-1} + \cdots + b_{N-1}s + b_N}{s^N + a_1 s^{N-1} + \cdots + a_{N-1}s + a_N}$$

Nth-order direct form II realization:



letting the N integrator outputs $q_1, q_2, \ldots, q_N$, we have

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_{N-1} \\ \dot{q}_N \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N-1} \\ q_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x$$

$$y = \begin{bmatrix} \hat{b}_N & \hat{b}_{N-1} & \cdots & \hat{b}_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix} + b_0 x$$

where $\hat{b}_i = b_i - b_0 a_i$

Example 11.4

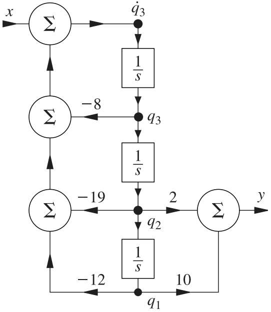
consider the system described by the transfer function

$$H(s) = \underbrace{\frac{2s+10}{s^3+8s^2+19s+12}}_{\text{direct form}}$$

$$= \underbrace{\left(\frac{2}{s+1}\right)\left(\frac{s+5}{s+3}\right)\left(\frac{1}{s+4}\right)}_{\text{cascade}} = \underbrace{\frac{\frac{4}{3}}{s+1} - \frac{2}{s+3} + \frac{\frac{2}{3}}{s+4}}_{\text{parallel}}$$

- we can realize this transfer function in (canonic) direct form II (DFII), transpose DFII (TDFII), cascade, and parallel realizations
- determine the state-space descriptions for each of these realizations

Solution: *direct form II*



we choose the state variables to be the three integrator outputs q_1, q_2, and q_3, then

$$\begin{aligned}
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= q_3 \\
\dot{q}_3 &= -12q_1 - 19q_2 - 8q_3 + x \\
y &= 10q_1 + 2q_2
\end{aligned}$$

in matrix form, these state and output equations become

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -19 & -8 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{B}} x$$

$$\boldsymbol{y} = \underbrace{\begin{bmatrix} 10 & 2 & 0 \end{bmatrix}}_{\boldsymbol{C}} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

- MATLAB's `tf2ss` command finds the state equations of the DFII structure:

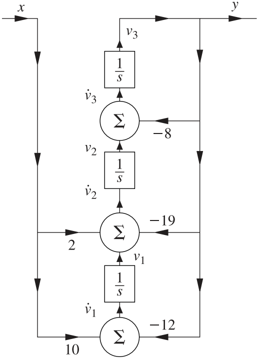
```
>> num = [2 10]; den = [1 8 19 12];
>> [A,B,C,D] = tf2ss(num,den)
A = -8 -19 -12
1 0 0
0 1 0
B = 1
0
0
C = 0 2 10
D = 0
```

  MATLAB labels q_1 as q_n, q_2 and q_{n-1}, and so on

- we can find the transfer function from the state-space representation using the `ss2tf` and `tf` commands:

```
>> [num,den] = ss2tf(A,B,C,D); H = tf(num,den)
H =
2 s + 10
-----------------------
s^3 + 8 s^2 + 19 s + 12
```

transpose direct form II



let the state variables v_1, v_2, and v_3 be the output of the three integrators, then

$$\begin{aligned}\dot{v}_1 &= -12v_3 + 10x\\ \dot{v}_2 &= v_1 - 19v_3 + 2x\\ \dot{v}_3 &= v_2 - 8v_3\\ y &= v_3\end{aligned}$$

the matrix form of these state and output equations are

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -12 \\ 1 & 0 & -19 \\ 0 & 1 & -8 \end{bmatrix}}_{\hat{\boldsymbol{A}}} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 10 \\ 2 \\ 0 \end{bmatrix}}_{\hat{\boldsymbol{B}}} x$$

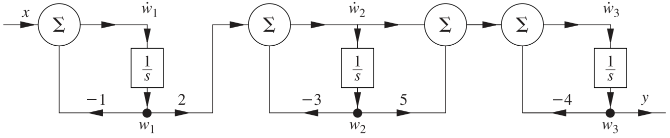
$$\boldsymbol{y} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{\hat{\boldsymbol{C}}} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

relationship between the state equation of the DFII and TDFII realizations

- $\boldsymbol{A}$ matrices in these two cases are the transpose of each other
- $\boldsymbol{B}$ of one is the transpose of $\boldsymbol{C}$ in the other, and vice versa

$$\boldsymbol{A}^\top = \hat{\boldsymbol{A}}, \quad \boldsymbol{B}^\top = \hat{\boldsymbol{C}}, \quad \boldsymbol{C}^\top = \hat{\boldsymbol{B}}$$

cascade realization:



let the three integrator outputs w_1, w_2, and w_3 be the state variables:

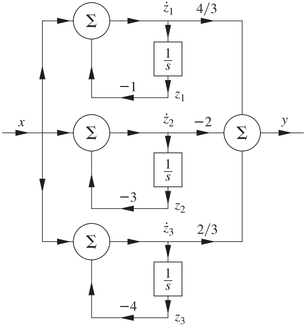
$$\dot{w}_1 = -w_1 + x, \quad \dot{w}_2 = 2w_1 - 3w_2, \quad \text{and} \quad \dot{w}_3 = 2w_1 + 2w_2 - 4w_3$$

we also have $y = w_3$; put into matrix form, we obtain

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

parallel realization:



the three integrator outputs z_1, z_2, and z_3 are the state variables:

$$\begin{aligned}\dot{z}_1 &= -z_1 + x \\ \dot{z}_2 &= -3z_2 + x \\ \dot{z}_3 &= -4z_3 + x \\ y &= \frac{4}{3}z_1 - 2z_2 + \frac{2}{3}z_3\end{aligned}$$

in matrix form, these equations are

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x$$

$$y = \begin{bmatrix} \frac{4}{3} & -2 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

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- state equations from transfer function
- **Laplace transform solution of state equations**
- state-equations of discrete-time systems

Linear system state-space equations

LTIC systems can be represented as

$$\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{q} + \boldsymbol{B}\boldsymbol{x}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{q} + \boldsymbol{D}\boldsymbol{x}$$

- $\boldsymbol{q} = (q_1, q_2, \ldots, q_N)$ is the *state vector*
- $\boldsymbol{x} = (x_1, x_2, \ldots, x_M)$ is the *input vector*
- $\boldsymbol{A}$ is an $N \times N$ matrix and $\boldsymbol{B}$ is an $N \times M$ matrix
- $\boldsymbol{y} = (y_1, y_2, \ldots, y_k)$ is the *output vector*
- $\boldsymbol{C}$ is an $k \times N$ matrix and $\boldsymbol{D}$ is an $k \times M$ matrix

Laplace transform solution of state equations

the ith state equation is of the form

$$\dot{q}_i = a_{i1}q_1 + a_{i2}q_2 + \cdots + a_{iN}q_N + b_{i1}x_1 + b_{i2}x_2 + \cdots + b_{ij}x_j$$

taking Laplace transform

$$\begin{aligned} sQ_i(s) - q_i(0) =& a_{i1}Q_1(s) + a_{i2}Q_2(s) + \cdots + a_{iN}Q_N(s) + b_{i1}X_1(s) \\ &+ b_{i2}X_2(s) + \cdots + b_{ij}X_j(s) \end{aligned}$$

taking the Laplace transforms of all N state equations, we obtain

$$s\underbrace{\begin{bmatrix} Q_1(s) \\ Q_2(s) \\ \vdots \\ Q_N(s) \end{bmatrix}}_{\boldsymbol{Q}(s)} - \underbrace{\begin{bmatrix} q_1(0) \\ q_2(0) \\ \vdots \\ q_N(0) \end{bmatrix}}_{\boldsymbol{q}(0)} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} Q_1(s) \\ Q_2(s) \\ \vdots \\ Q_N(s) \end{bmatrix}}_{\boldsymbol{Q}(s)}$$

$$+ \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} \\ b_{21} & b_{22} & \cdots & b_{2j} \\ \vdots & \vdots & \cdots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{Nj} \end{bmatrix}}_{\boldsymbol{B}} \underbrace{\begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_j(s) \end{bmatrix}}_{\boldsymbol{X}(s)}$$

or

$$s\boldsymbol{Q}(s) - \boldsymbol{q}(0) = \boldsymbol{A}\boldsymbol{Q}(s) + \boldsymbol{B}\boldsymbol{X}(s) \Rightarrow (s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{Q}(s) = \boldsymbol{q}(0) + \boldsymbol{B}\boldsymbol{X}(s)$$

where $\boldsymbol{I}$ is the $N \times N$ identity matrix

Solution of state space equations

Laplace solution of state-space equations

$$\begin{aligned}\boldsymbol{Q}(s) &= (s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{q}(0) + \boldsymbol{B}\boldsymbol{X}(s)] \\ &= \boldsymbol{\Phi}(s)[\boldsymbol{q}(0) + \boldsymbol{B}\boldsymbol{X}(s)]\end{aligned}$$

with

$$\boldsymbol{\Phi}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}$$

Solution of state-space equations

$$\boldsymbol{q}(t) = \underbrace{\mathcal{L}^{-1}[\boldsymbol{\Phi}(s)]\boldsymbol{q}(0)}_{\text{zero-input response}} + \underbrace{\mathcal{L}^{-1}[\boldsymbol{\Phi}(s)\boldsymbol{B}\boldsymbol{X}(s)]}_{\text{zero-state response}}$$

Example 11.5

find the state vector $\boldsymbol{q}(t)$ for the system

$$\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{q} + \boldsymbol{B}\boldsymbol{x}$$

where

$$\boldsymbol{A} = \begin{bmatrix} -12 & \frac{2}{3} \\ -36 & -1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \quad \boldsymbol{x}(t) = u(t)$$

and the initial conditions are $q_1(0) = 2$, $q_2(0) = 1$

Solution: we have

$$\boldsymbol{Q}(s) = \boldsymbol{\Phi}(s)[\boldsymbol{q}(0) + \boldsymbol{B}\boldsymbol{X}(s)]$$

let us first find $\boldsymbol{\Phi}(s)$, we have

$$(s\boldsymbol{I} - \boldsymbol{A}) = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -12 & \frac{2}{3} \\ -36 & -1 \end{bmatrix} = \begin{bmatrix} s+12 & -\frac{2}{3} \\ 36 & s+1 \end{bmatrix}$$

and

$$\boldsymbol{\Phi}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1} = \begin{bmatrix} \frac{s+1}{(s+4)(s+9)} & \frac{2/3}{(s+4)(s+9)} \\ \frac{-36}{(s+4)(s+9)} & \frac{s+12}{(s+4)(s+9)} \end{bmatrix}$$

now, $\boldsymbol{q}(0)$ is given as

$$\boldsymbol{q}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

also, $\boldsymbol{X}(s) = 1/s$, and

$$\boldsymbol{B}\boldsymbol{X}(s) = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \frac{1}{s} = \begin{bmatrix} \frac{1}{3s} \\ \frac{1}{s} \end{bmatrix}$$

therefore,

$$\boldsymbol{q}(0) + \boldsymbol{B}\boldsymbol{X}(s) = \begin{bmatrix} 2 + \frac{1}{3s} \\ 1 + \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{6s+1}{3s} \\ \frac{s+1}{s} \end{bmatrix}$$

and

$$
\begin{aligned}
\boldsymbol{Q}(s) &= \boldsymbol{\Phi}(s)[\boldsymbol{q}(0) + \boldsymbol{B}\boldsymbol{X}(s)] \\
&= \begin{bmatrix} \frac{s+1}{(s+4)(s+9)} & \frac{2/3}{(s+4)(s+9)} \\ \frac{-36}{(s+4)(s+9)} & \frac{s+12}{(s+4)(s+9)} \end{bmatrix} \begin{bmatrix} \frac{6s+1}{3s} \\ \frac{s+1}{s} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2s^2+3s+1}{s(s+4)(s+9)} \\ \frac{s-59}{(s+4)(s+9)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1/36}{s} - \frac{21/20}{s+4} + \frac{136/45}{s+9} \\ \frac{-63/5}{s+4} + \frac{68/5}{s+9} \end{bmatrix}
\end{aligned}
$$

the inverse Laplace transform of this equation yields

$$
\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{36} - \frac{21}{20}e^{-4t} + \frac{136}{45}e^{-9t}\right)u(t) \\ \left(-\frac{63}{5}e^{-4t} + \frac{68}{5}e^{-9t}\right)u(t) \end{bmatrix}
$$

The output

the output equation is given by

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{q} + \boldsymbol{D}\boldsymbol{x}$$

taking Laplace transform

$$\boldsymbol{Y}(s) = \boldsymbol{C}\boldsymbol{Q}(s) + \boldsymbol{D}\boldsymbol{X}(s)$$

hence,

$$\begin{aligned} \boldsymbol{Y}(s) &= \boldsymbol{C}\{\boldsymbol{\Phi}(s)[\boldsymbol{q}(0) + \boldsymbol{B}\boldsymbol{X}(s)]\} + \boldsymbol{D}\boldsymbol{X}(s) \\ &= \underbrace{\boldsymbol{C}\boldsymbol{\Phi}(s)\boldsymbol{q}(0)}_{\text{zero-input response}} + \underbrace{[\boldsymbol{C}\boldsymbol{\Phi}(s)\boldsymbol{B} + \boldsymbol{D}]\boldsymbol{X}(s)}_{\text{zero-state response}} \end{aligned}$$

the zero-state response [i.e., the response $\boldsymbol{Y}(s)$ when $\boldsymbol{q}(0) = \mathbf{0}$] is given by

$$\boldsymbol{Y}(s) = [\boldsymbol{C}\boldsymbol{\Phi}(s)\boldsymbol{B} + \boldsymbol{D}]\boldsymbol{X}(s)$$

Transfer function

the transfer function matrix is:

$$\boldsymbol{H}(s) = \boldsymbol{C}\boldsymbol{\Phi}(s)\boldsymbol{B} + \boldsymbol{D}$$

and the zero-state response is

$$\boldsymbol{Y}(s) = \boldsymbol{H}(s)\boldsymbol{X}(s)$$

- $\boldsymbol{H}(s)$ is a $k \times j$ matrix (k is the number of outputs and j is the number of inputs)
- $H_{ij}(s)$ is the transfer function that relates the output $y_i(t)$ to the input $x_j(t)$

Example 11.6

consider a system with a state equation

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and an output equation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

determine the transfer function matrix of the system

Solution: here

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \boldsymbol{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\boldsymbol{\Phi}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

hence, the transfer function matrix $\boldsymbol{H}(s)$ is given by

$$
\begin{aligned}
\boldsymbol{H}(s) &= \boldsymbol{C}\boldsymbol{\Phi}(s)\boldsymbol{B} + \boldsymbol{D} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{s+4}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{s+4}{s+2} & \frac{1}{s+2} \\ \frac{2(s-2)}{(s+1)(s+2)} & \frac{s^2+5s+2}{(s+1)(s+2)} \end{bmatrix}
\end{aligned}
$$

and the zero-state response is

$$\boldsymbol{Y}(s) = \boldsymbol{H}(s)\boldsymbol{X}(s)$$

the transfer function that relates the output y_3 to the input x_2 is

$$H_{32}(s) = \frac{s^2 + 5s + 2}{(s+1)(s+2)}$$

Outline

- state-space modeling
- state equations from transfer function
- Laplace transform solution of state equations
- **state-equations of discrete-time systems**

State-equation of discrete-time system

an Nth-order difference equation can be expressed in terms of N first-order difference (state) equations

LTID systems state-space equation has the form:

$$\begin{aligned} \boldsymbol{q}[n+1] &= \boldsymbol{A}\boldsymbol{q}[n] + \boldsymbol{B}\boldsymbol{x}[n] \\ \boldsymbol{y}[n] &= \boldsymbol{C}\boldsymbol{q}[n] + \boldsymbol{D}\boldsymbol{x}[n] \end{aligned}$$

- $\boldsymbol{q} = (q_1, q_2, \ldots, q_N)$ is the *state vector*
- $\boldsymbol{x} = (x_1, x_2, \ldots, x_M)$ is the *input vector*
- $\boldsymbol{A}$ is an $N \times N$ matrix; $\boldsymbol{B}$ is an $N \times M$ matrix
- $\boldsymbol{y} = (y_1, y_2, \ldots, y_k)$ is the *output vector*
- $\boldsymbol{C}$ is an $k \times N$ matrix; $\boldsymbol{D}$ is an $k \times M$ matrix

DFII state-space description

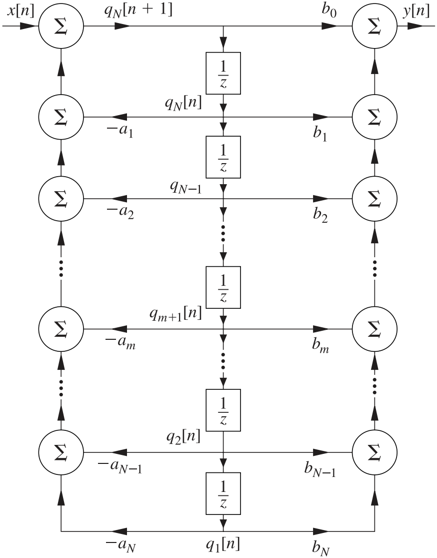
consider the z-transfer function

$$H[z] = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1} z + b_N}{z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}$$

the input $x[n]$ and the output $y[n]$ of this system are related by

$$\begin{aligned}(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y[n] \\ = (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N) x[n]\end{aligned}$$

DFII realization



we can write N equations, one at the input of each delay and output equation:

$$\begin{aligned} q_1[n+1] &= q_2[n] \\ q_2[n+1] &= q_3[n] \\ &\vdots \\ q_{N-1}[n+1] &= q_N[n] \\ q_N[n+1] &= -a_N q_1[n] - a_{N-1} q_2[n] - \cdots - a_1 q_N[n] + x[n] \\ y[n] &= b_N q_1[n] + b_{N-1} q_2[n] + \cdots + b_1 q_N[n] + b_0 q_{N+1}[n] \end{aligned}$$

eliminating $q_{N+1}[n]$ from the output equation gives

$$y[n] = \hat{b}_N q_1[n] + \hat{b}_{N-1} q_2[n] + \cdots + \hat{b}_1 q_N[n] + b_0 x[n]$$

where $\hat{b}_i = b_i - b_0 a_i$

DFII discrete-time state equation in matrix form

$$\underbrace{\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \\ \vdots \\ q_{N-1}[n+1] \\ q_N[n+1] \end{bmatrix}}_{\boldsymbol{q}[n+1]} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_2 & -a_1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} q_1[n] \\ q_2[n] \\ \vdots \\ q_{N-1}[n] \\ q_N[n] \end{bmatrix}}_{\boldsymbol{q}[n]}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{B}} x[n]$$

$$\boldsymbol{y}[n] = \underbrace{\begin{bmatrix} \hat{b}_N & \hat{b}_{N-1} & \cdots & \hat{b}_1 \end{bmatrix}}_{\boldsymbol{C}} \begin{bmatrix} q_1[n] \\ q_2[n] \\ \vdots \\ q_N[n] \end{bmatrix} + \underbrace{b_0}_{\boldsymbol{D}} x[n]$$

Recursive solution of discrete-time state equations

consider the state equation

$$\boldsymbol{q}[n+1] = \boldsymbol{A}\boldsymbol{q}[n] + \boldsymbol{B}\boldsymbol{x}[n]$$

from this, we have

$$\begin{aligned}
\boldsymbol{q}[n] &= \boldsymbol{A}\boldsymbol{q}[n-1] + \boldsymbol{B}\boldsymbol{x}[n-1] \\
\boldsymbol{q}[n-1] &= \boldsymbol{A}\boldsymbol{q}[n-2] + \boldsymbol{B}\boldsymbol{x}[n-2] \\
\boldsymbol{q}[n-2] &= \boldsymbol{A}\boldsymbol{q}[n-3] + \boldsymbol{B}\boldsymbol{x}[n-3] \\
&\vdots \\
\boldsymbol{q}[1] &= \boldsymbol{A}\boldsymbol{q}[0] + \boldsymbol{B}\boldsymbol{x}[0]
\end{aligned}$$

substituting the expression for $\boldsymbol{q}[n-1]$ into that for $\boldsymbol{q}[n]$, we obtain

$$\boldsymbol{q}[n] = \boldsymbol{A}^2\boldsymbol{q}[n-2] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{x}[n-2] + \boldsymbol{B}\boldsymbol{x}[n-1]$$

substituting the expression for $\boldsymbol{q}[n-2]$ in this equation, we obtain

$$\boldsymbol{q}[n] = \boldsymbol{A}^3\boldsymbol{q}[n-3] + \boldsymbol{A}^2\boldsymbol{B}\boldsymbol{x}[n-3] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{x}[n-2] + \boldsymbol{B}\boldsymbol{x}[n-1]$$

continuing in this way, we obtain

$$\begin{aligned}\boldsymbol{q}[n] &= \boldsymbol{A}^n\boldsymbol{q}[0] + \boldsymbol{A}^{n-1}\boldsymbol{B}\boldsymbol{x}[0] + \boldsymbol{A}^{n-2}\boldsymbol{B}\boldsymbol{x}[1] + \cdots + \boldsymbol{B}\boldsymbol{x}[n-1] \\ &= \boldsymbol{A}^n\boldsymbol{q}[0] + \sum_{m=0}^{n-1}\boldsymbol{A}^{n-1-m}\boldsymbol{B}\boldsymbol{x}[m]\end{aligned}$$

thus

$$\begin{aligned}\boldsymbol{y}[n] &= \boldsymbol{C}\boldsymbol{q} + \boldsymbol{D}\boldsymbol{x} \\ &= \boldsymbol{C}\boldsymbol{A}^n\boldsymbol{q}[0] + \sum_{m=0}^{n-1}\boldsymbol{C}\boldsymbol{A}^{n-1-m}\boldsymbol{B}\boldsymbol{x}[m] + \boldsymbol{D}\boldsymbol{x}\end{aligned}$$

References

- B. P. Lathi, R. A. Green. *Linear Systems and Signals*. Oxford University Press, 2018.