10. Analysis using z-transform

- solution of linear difference equations
- the transfer function
- frequency response
- realization of DT systems

Solving linear difference equations

- the z-transform converts difference equations into algebraic equations
- taking the inverse z-transform of the z-domain solution yields the desired solution

Example:

$$y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n]$$

with $y[-1] = 11/6, y[-2] = 37/36$, and input $x[n] = (2)^{-n}u[n]$

- using left-shift property requires a knowledge of conditions $y[0], \ldots, y[N-1]$
- to directly use initial conditions, we express the difference equation in delay form and use the right-shift property

the delay-form difference equation is

$$y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2]$$

here, $y[n-m]$ (or $x[n-m]$) means $y[n-m]u[n]$ (or $x[n-m]u[n]$); we have

$$\begin{aligned}
y[n]u[n] &\Longleftrightarrow Y(z) \\
y[n-1]u[n] &\Longleftrightarrow \frac{1}{z}Y(z) + y[-1] = \frac{1}{z}Y(z) + \frac{11}{6} \\
y[n-2]u[n] &\Longleftrightarrow \frac{1}{z^2}Y(z) + \frac{1}{z}y[-1] + y[-2] = \frac{1}{z^2}Y(z) + \frac{11}{6z} + \frac{37}{36}
\end{aligned}$$

since $x[n]$ is causal, $x[n-m]u[n] \Longleftrightarrow \frac{1}{z^m}X(z)$, we thus have

$$\begin{aligned}
x[n] = (2)^{-n}u[n] = (0.5)^n u[n] &\Longleftrightarrow \frac{z}{z-0.5} \\
x[n-1]u[n] &\Longleftrightarrow \frac{1}{z}X(z) = \frac{1}{z}\frac{z}{z-0.5} = \frac{1}{z-0.5} \\
x[n-2]u[n] &\Longleftrightarrow \frac{1}{z^2}X(z) = \frac{1}{z^2}X(z) = \frac{1}{z(z-0.5)}
\end{aligned}$$

taking the z-transform of the difference equation:

$$Y(z) - 5\left[\frac{1}{z}Y(z) + \frac{11}{6}\right] + 6\left[\frac{1}{z^2}Y(z) + \frac{11}{6z} + \frac{37}{36}\right] = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y(z) - \left(3 - \frac{11}{z}\right) = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

rearranging gives,

$$\frac{Y(z)}{z} = \frac{3z^2 - 9.5z + 10.5}{(z-0.5)(z-2)(z-3)} = \frac{(26/15)}{z-0.5} - \frac{(7/3)}{z-2} + \frac{(18/5)}{z-3}$$

therefore,

$$Y(z) = \tfrac{26}{15}\left(\frac{z}{z-0.5}\right) - \tfrac{7}{3}\left(\frac{z}{z-2}\right) + \tfrac{18}{5}\left(\frac{z}{z-3}\right)$$

and

$$y[n] = \left[\tfrac{26}{15}(0.5)^n - \tfrac{7}{3}(2)^n + \tfrac{18}{5}(3)^n\right]u[n]$$

Zero-input and zero-state components

- we can separate the solution into zero-input and zero-state components
- separate the response into terms arising from the input and terms arising from i.c.

in the previous example, we have

$$\left(1-\frac{5}{z}+\frac{6}{z^2}\right)Y(z)=\underbrace{\left(3-\frac{11}{z}\right)}_{\text{IC terms}}+\underbrace{\frac{(3z+5)}{z(z-0.5)}}_{\text{input terms}}$$

multiplying both sides by z^2 yields

$$\left(z^2-5z+6\right)Y(z)=\underbrace{z(3z-11)}_{\text{IC terms}}+\underbrace{\frac{z(3z+5)}{z-0.5}}_{\text{input terms}}$$

hence,

$$Y(z) = \underbrace{\frac{z(3z-11)}{z^2-5z+6}}_{\text{zero-input response}} + \underbrace{\frac{z(3z+5)}{(z-0.5)\,(z^2-5z+6)}}_{\text{zero-state response}}$$

we expand both terms on the right-hand side into modified partial fractions:

$$Y(z) = \underbrace{\left[5\left(\frac{z}{z-2}\right) - 2\left(\frac{z}{z-3}\right)\right]}_{\text{zero-input response}} + \underbrace{\left[\frac{26}{15}\left(\frac{z}{z-0.5}\right) - \frac{22}{3}\left(\frac{z}{z-2}\right) + \frac{28}{5}\left(\frac{z}{z-3}\right)\right]}_{\text{zero-state response}}$$

thus

$$y[n] = \underbrace{(5(2)^n - 2(3)^n)\,u[n]}_{\text{zero-input response}} + \underbrace{\left(\frac{26}{15}(0.5)^n - \frac{22}{3}(2)^n + \frac{28}{5}(3)^n\right)u[n]}_{\text{zero-state response}}$$

$$= \left[-\frac{7}{3}(2)^n + \frac{18}{5}(3)^n + \frac{26}{15}(0.5)^n\right]u[n]$$

Outline

- solution of linear difference equations
- **the transfer function**
- frequency response
- realization of DT systems

The transfer function

the *transfer function* of an LTID system with impulse response $h[n]$ is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- $H(z)$ is z-transform of impulse response $h[n]$
- the LTID system response $y[n]$ to an everlasting exponential z^n is

$$y[n] = h[n] * z^n = \sum_{m=-\infty}^{\infty} h[m] z^{n-m} = H(z) z^n$$

  for fixed z, the output $y[n] = H(z)z^n$ has same form as input z^n
  – this input is called *eigenfunction*
- an alternate definition of the transfer function $H(z)$ of an LTID system is

$$H(z) = \left. \frac{\text{output signal}}{\text{input signal}} \right|_{\text{input=exponential } z^n}$$

Zero-state response

taking z-transform of $y[n] = x[n] * h[n]$, we have

$$Y(z) = X(z)H(z)$$

- we can find zero state response by taking the inverse z-transform:

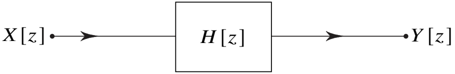
$$y[n] = \mathcal{Z}^{-1}\{X(z)H(z)\}$$

- given the input and output, we can find transfer function as

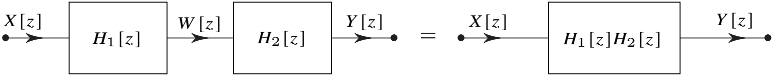
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\mathcal{Z}[\text{zero-state response}]}{\mathcal{Z}[\text{input}]}$$

Block diagrams

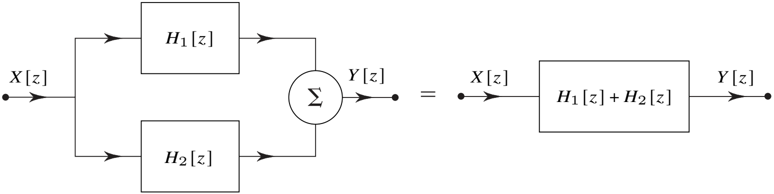
Block diagram of linear system



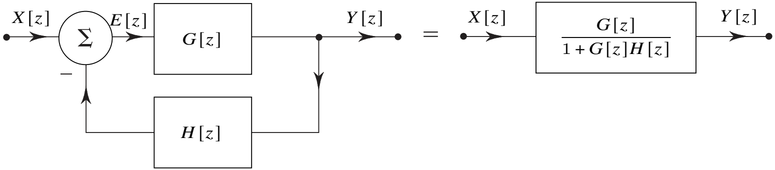
Cascade interconnection



Parallel interconnection



Feedback interconnection



$$\frac{Y(z)}{X(z)} = \frac{G(z)}{1 + G(z)H(z)}$$

Unit delay: the unit delay, which is represented by a box marked D, will be represented by its transfer function $1/z$

Transfer function of LTI difference system

Nth-order LTID system

$$Q[E]y[n] = P[E]x[n]$$

or

$$\begin{aligned}(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N)y[n] \\ = (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1}E + b_N)x[n]\end{aligned}$$

the transfer function is

$$H(z) = \frac{P(z)}{Q(z)} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1}z + b_N}{z^N + a_1 z^{N-1} + \cdots + a_{N-1}z + a_N}$$

Example 10.1

consider an LTID system described by the difference equation

$$y[n+2] + y[n+1] + 0.16y[n] = x[n+1] + 0.32x[n]$$

or

$$\left(E^2 + E + 0.16\right) y[n] = (E + 0.32)x[n]$$

find the transfer function and the zero-state response $y[n]$ if $x[n] = (-2)^{-n}u[n]$

Solution: from the difference equation, we find

$$H(z) = \frac{P(z)}{Q(z)} = \frac{z + 0.32}{z^2 + z + 0.16}$$

the input $x[n] = (-2)^{-n}u[n] = (-0.5)^n u[n]$ z-transform is

$$X(z) = \frac{z}{z + 0.5}$$

therefore,

$$Y(z) = X(z)H(z) = \frac{z(z+0.32)}{(z^2+z+0.16)\,(z+0.5)}$$

and

$$\begin{aligned}\frac{Y(z)}{z} &= \frac{(z+0.32)}{(z^2+z+0.16)\,(z+0.5)} = \frac{(z+0.32)}{(z+0.2)(z+0.8)(z+0.5)} \\ &= \frac{2/3}{z+0.2} - \frac{8/3}{z+0.8} + \frac{2}{z+0.5}\end{aligned}$$

so that

$$Y(z) = \frac{2}{3}\left(\frac{z}{z+0.2}\right) - \frac{8}{3}\left(\frac{z}{z+0.8}\right) + 2\left(\frac{z}{z+0.5}\right)$$

and

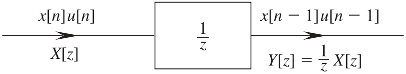
$$y[n] = \left[\frac{2}{3}(-0.2)^n - \frac{8}{3}(-0.8)^n + 2(-0.5)^n\right] u[n]$$

Example 10.2

if the input to the unit delay is $x[n]u[n]$, then its output is given by

$$y[n] = x[n-1]u[n-1]$$

show that the transfer function of a unit delay is $1/z$



Solution: the z-transform of this equation yields

$$Y(z) = \frac{1}{z}X(z) = H(z)X(z)$$

it follows that the transfer function of the unit delay is

$$H(z) = \frac{1}{z}$$

Stability

BIBO stability

- if all the poles of $H(z)$ are within the unit circle, then system is BIBO-stable
  - all the terms in $h[n]$ are decaying exponentials and $h[n]$ is absolutely summable
- otherwise the system is BIBO-unstable

Internal stability: if $P(z)$ and $Q(z)$ have **no** common factors, then the poles of $H(z)$ are the characteristic roots of the system; hence an LTID system is

1. asymptotically stable if and only if all the poles are within the unit circle
2. unstable if and only if either one or both of the following conditions exist:
   - (i) at least one pole of $H(z)$ is outside the unit circle
   - (ii) there are repeated poles of $H(z)$ on the unit circle
3. marginally stable if and only if there are no poles outside the unit circle, and there are some simple poles on the unit circle

Inverse systems

if $H(z)$ is the transfer function of a system $\mathcal{S}$, then $\mathcal{S}_i$, its inverse system, has a transfer function $H_i(z)$ given by

$$H_i(z) = \frac{1}{H(z)}$$

Examples:

- an accumulator $H(z) = z/(z-1)$ and a backward difference system $H_i(z) = (z-1)/z$ are inverse of each other
- if

  $$H(z) = \frac{z-0.4}{z-0.7}$$

  its inverse system transfer function is

  $$H_i(z) = \frac{z-0.7}{z-0.4}$$

  as required by the property $H(z)H_i(z) = 1$; hence, it follows that

  $$h[n] * h_i[n] = \delta[n]$$

Outline

- solution of linear difference equations
- the transfer function
- **frequency response**
- realization of DT systems

Frequency response

the LTID system response to complex sinusoid $x[n] = A_x e^{j\Omega n}$ is

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] A_x \, e^{j\Omega(n-m)} = H(e^{j\Omega}) A_x e^{j\Omega n}$$

$$= |H(e^{j\Omega})||A_x| e^{j\left(\Omega n + \angle H(e^{j\Omega}) + \angle A_x\right)}$$

- response is also complex sinusoid of the same frequency Ω multiplied by $H(e^{j\Omega})$
- $H(e^{j\Omega})$ is the *frequency response* of the system
- using frequency response, we can find output for any sinusoidal input
- $H(e^{j\Omega})$ is a periodic function of Ω with period 2π since

$$H(e^{j\Omega}) = H(e^{j(\Omega+2\pi m)}) \quad m \text{ integer}$$

Sinusoidal input response: the response to $\mathrm{Re}(e^{j(\Omega n+\theta)}) = \cos(\Omega n + \theta)$ is

$$y[n] = |H(e^{j\Omega})| \cos\left(\Omega n + \theta + \angle H(e^{j\Omega})\right)$$

Amplitude response

- $|H(e^{j\Omega})|$ is the amplitude gain called *amplitude response* or *magnitude response*
- plot $|H(e^{j\Omega})|$ v Ω shows the amplitude gain as a function of frequency Ω

Phase response

- $\angle H(e^{j\Omega})$ is the *phase response*
- plot $\angle H(e^{j\Omega})$ v Ω shows how the system changes the phase of the input sinusoid

Steady-state response to causal inputs

- the response of an LTID system to a causal sinusoidal input $\cos(\Omega n)u[n]$ is $y[n]$, plus a natural component consisting of the characteristic modes
- for a stable system, the steady-state response of a system to a causal sinusoidal input $x[n] = \cos(\Omega n)u[n]$ is

$$y_{ss}[n] = \left|H(e^{j\Omega})\right| \cos\left(\Omega n + \angle H(e^{j\Omega})\right)$$

Response to sampled CT sinusoids

- input may be a sampled CT sinusoid $\cos \omega t$ (or an exponential $e^{j\omega t}$)
- $\cos \omega t$ sampled with sampling interval T is DT sinusoid $\cos \omega nT$
- therefore, all the results developed here apply if we substitute ωT for Ω:

$$\Omega = \omega T$$

Example 10.3

for a system described by the equation

$$y[n+1] - 0.8y[n] = x[n+1]$$

find the system response to the inputs

(a) $x[n] = \cos(\frac{\pi}{6}n - 0.2)$

(b) $x[n] = 1$

(c) a sampled sinusoid $\cos(1500t)$ with sampling interval $T = 0.001$

Solution: the system equation can be expressed as

$$(E - 0.8)y[n] = Ex[n]$$

the transfer function of the system is

$$H(z) = \frac{z}{z - 0.8} = \frac{1}{1 - 0.8z^{-1}}$$

the frequency response is

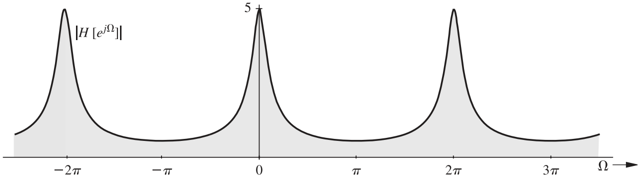
$$H(e^{j\Omega}) = \frac{1}{1 - 0.8e^{-j\Omega}} = \frac{1}{(1 - 0.8\cos\Omega) + j0.8\sin\Omega}$$

therefore,

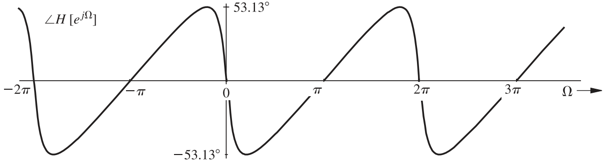
$$\left|H(e^{j\Omega})\right| = \frac{1}{\sqrt{(1 - 0.8\cos\Omega)^2 + (0.8\sin\Omega)^2}} = \frac{1}{\sqrt{1.64 - 1.6\cos\Omega}}$$

and

$$\angle H(e^{j\Omega}) = -\tan^{-1}\left[\frac{0.8\sin\Omega}{1 - 0.8\cos\Omega}\right]$$


5
$|H[e^{j\Omega}]|$
-2π
$-\pi$
0
π
2π
3π
Ω

(a)


53.13°
$\angle H[e^{j\Omega}]$
-2π
$-\pi$
0
π
2π
3π
Ω
−53.13°

(b)

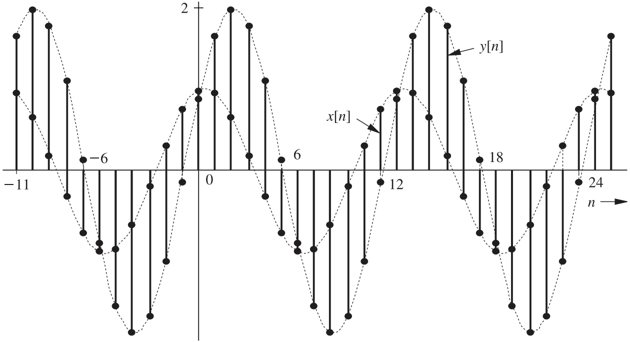
(a) for $x[n] = \cos[(\pi/6)n - 0.2], \Omega = \pi/6$ and

$$\left|H\left(e^{j\pi/6}\right)\right| = \frac{1}{\sqrt{1.64 - 1.6\cos\frac{\pi}{6}}} = 1.983$$

$$\angle H\left(e^{j\pi/6}\right) = -\tan^{-1}\left[\frac{0.8\sin\frac{\pi}{6}}{1 - 0.8\cos\frac{\pi}{6}}\right] = -0.916\,\mathrm{rad}$$

therefore,

$$y[n] = 1.983\cos\left(\frac{\pi}{6}n - 0.2 - 0.916\right) = 1.983\cos\left(\frac{\pi}{6}n - 1.116\right)$$



(b) since $1^n = (e^{j\Omega})^n$ with $\Omega = 0$, the amplitude response is

$$H\left(e^{j0}\right) = \frac{1}{\sqrt{1.64 - 1.6\cos(0)}} = \frac{1}{\sqrt{0.04}} = 5 = 5\angle 0$$

therefore,

$$\left|H\left(e^{j0}\right)\right| = 5 \quad \text{and} \quad \angle H\left(e^{j0}\right) = 0$$

and the system response to input 1 is

$$y[n] = 5\left(1^n\right) = 5 \quad \text{for all } n$$

(c) sampling $\cos 1500t$ every $T = 0.001$, the input is

$$x[n] = \cos(1.5n)$$

in this case, $\Omega = 1.5$ and

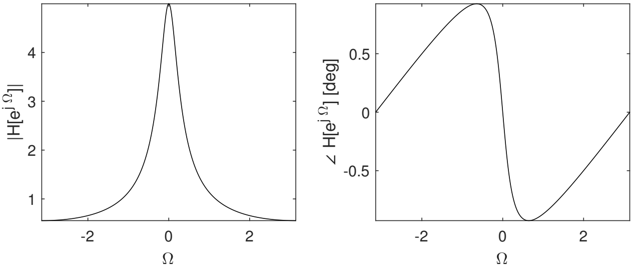
$$\left|H\left(e^{j1.5}\right)\right| = \frac{1}{\sqrt{1.64 - 1.6\cos(1.5)}} = 0.809$$
$$\angle H\left(e^{j1.5}\right) = -\tan^{-1}\left[\frac{0.8\sin(1.5)}{1 - 0.8\cos(1.5)}\right] = -0.702\mathrm{rad}$$

therefore,

$$y[n] = 0.809\cos(1.5n - 0.702)$$

Frequency response using MATLAB

```
Omega = linspace(-pi,pi,400); H = @(z) z./(z-0.8);
subplot(1,2,1); plot(Omega,abs(H(exp(1j*Omega))),'k'); axis tight;
xlabel('\Omega'); ylabel('|H(e^{j \Omega})|');
subplot(1,2,2); plot(Omega,angle(H(exp(1j*Omega))*180/pi),'k');
axis tight;
xlabel('\Omega'); ylabel('\angle H(e^{j \Omega}) [deg]');
```



Outline

- solution of linear difference equations
- the transfer function
- frequency response
- **realization of DT systems**

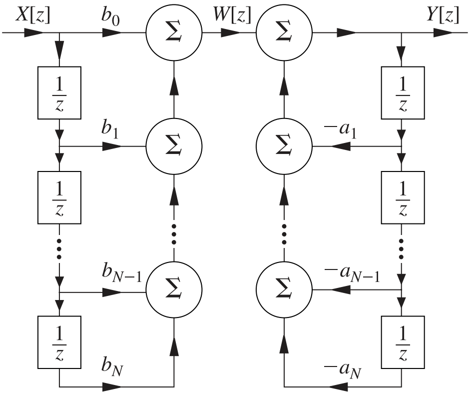
DT systems realization

we shall consider a realization of a general Nth-order causal LTID system,

$$H(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1} z + b_N}{z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}$$

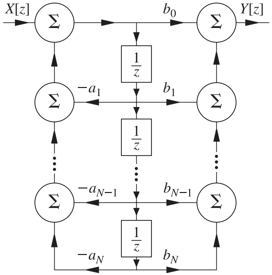
- procedure for realizing an LTID transfer function is identical to that for the LTIC system
- the basic element $1/s$ (integrator) replaced by the element $1/z$ (unit delay)

Direct form I (DFI)

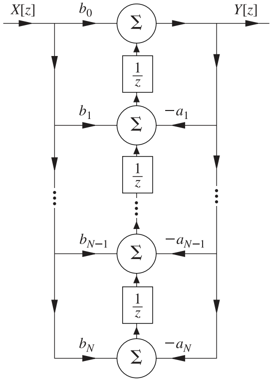


Canonic form

Canonic direct form (DFII)



Transpose of canonic direct form



the DFII and its transpose are canonic because they require N delays, which is the minimum number needed to implement the Nth-order LTID transfer function

Example 10.4

find the canonic direct and the transposed canonic direct realizations of the following:

(a) $\dfrac{2}{z+5}$

(b) $\dfrac{4z+28}{z+1}$

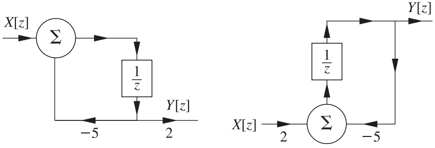
(c) $\dfrac{z}{z+7}$

(d) $\dfrac{4z+28}{z^2+6z+5}$

Solution:

(a) for this case, the transfer function is of the first order $(N = 1)$; therefore, we need only one delay for its realization

$$a_1 = 5 \quad \text{and} \quad b_0 = 0, \quad b_1 = 2$$

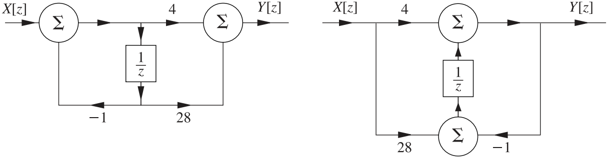


(a) (b)

canonic direct form and its transpose are shown above

(b) feedback and feedforward coefficients are

$$a_1 = 1 \quad \text{and} \quad b_0 = 4, \quad b_1 = 28$$



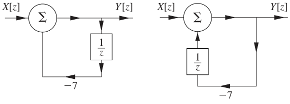
(a) (b)

transfer functions with $N = M$ may also be expressed as a sum of a constant and a strictly proper transfer function; for example,

$$H(z) = \frac{4z+28}{z+1} = 4 + \frac{24}{z+1}$$

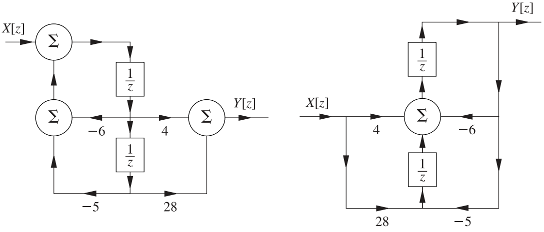
hence, this can also be realized as two transfer functions in parallel

(c) here, $N = 1$ and $b_0 = 1, b_1 = 0$ and $a_1 = 7$



(a) (b)

(d) here, $N = 2$ with $b_0 = 0, b_1 = 4, b_2 = 28, a_1 = 6, a_2 = 5$



(a) (b)

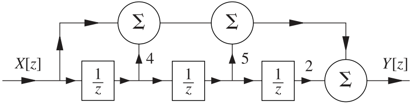
Realization of an FIR filter

- for *finite impulse response* (FIR) filters, the coefficients $a_i = 0$ for all $i \neq 0$
- FIR filters can be implemented by means of the schemes developed so far by eliminating all branches with a_i coefficients
- the condition $a_i = 0$ implies that all the poles of a FIR filter are at $z = 0$

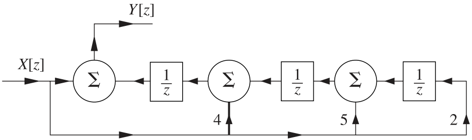
Example: let us realize

$$H(z) = \frac{z^3 + 4z^2 + 5z + 2}{z^3}$$

using canonic direct and transposed forms we have $b_0 = 1, b_1 = 4, b_2 = 5$, and $b_3 = 2$



(a)



(b)

this filter is basically a tapped delay line; it is also known as a tapped delay line or transversal filter

Cascade and parallel realizations, complex, and repeated poles: the considerations and observations for cascade and parallel realizations as well as complex and multiple poles are identical to those discussed for LTIC systems

References

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press.