Signals and System: Background

- complex numbers
- sinusoids and exponentials
- vectors and matrices
- matrix calculus
- partial fraction expansion

Complex numbers

Rectangular (Cartesian) form

$$z = a + jb$$

- number a is the *real part* of z denoted by $\mathrm{Re}\, z = a$
- number b is the *imaginary part* of z denoted by $\mathrm{Im}\, z = b$
- j is the *imaginary number:* $j^2 = -1$ and $\sqrt{-1} = \pm j$

Polar form

$$z = re^{j\theta} = r\underline{/\theta}$$

- $r = |z| > 0$ is the *magnitude* or *absolute value* of z
- θ is the *angle* of z
- $\underline{/\theta} = e^{j\theta}$

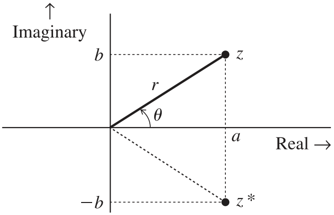
Rectangular and polar forms relation

using *Euler's formula* $e^{j\theta} = \cos\theta + j\sin\theta$, we have

$$z = re^{j\theta} = a + jb$$

$a = r\cos\theta$ and $b = r\sin\theta$

$r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(\frac{b}{a})$



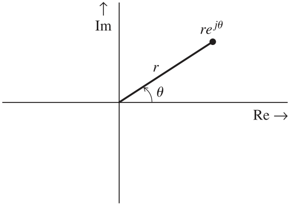
to use $\tan^{-1}(\frac{b}{a})$, proper attention must be taken to the quadrant:

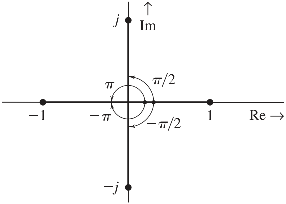
- $a > 0$ and $b > 0$ is in first quadrant: $0 < \theta < 90^\circ$
- $a < 0$ and $b > 0$ is in second quadrant: $90^\circ < \theta < 180^\circ$
- $a < 0$ and $b < 0$ is in third quadrant: $180^\circ < \theta < 270^\circ$
- $a > 0$ and $b < 0$ is in fourth quadrant: $270^\circ < \theta < 360^\circ$
- any angle θ is equivalent to $\theta \pm 360^\circ$

Useful identities

$$j = \frac{1}{-j}, \qquad e^{\pm j\pi/2} = \pm j, \qquad e^{(\alpha + j\omega)t} = e^{\alpha t} e^{j\omega t}$$

$$e^{\pm j2\pi n} = 1, \qquad e^{\pm j\pi + j2\pi n} = -1, \quad (n \text{ integer})$$





Complex numbers operations

let $z_1 = a_1 + jb_1 = r_1 e^{j\theta_1}$ and $z_2 = a_2 + jb_2 = re^{j\theta_2}$, then

Addition

$$z_1 + z_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

we need to convert to rectangular form to add complex numbers

Multiplication

$$z_1 z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

the reciprocal of a complex number is given by $\dfrac{1}{z} = \dfrac{1}{r} e^{-j\theta}$

let $z = a + jb = re^{j\theta}$

Complex conjugate

$$z^* = a - jb = re^{-j\theta}$$

note that $zz^* = z^*z = |z|^2$

Powers and roots

$$z^k = r^k e^{jk\theta}$$
$$z^{1/k} = r^{1/k} e^{j\theta/k}$$

- there are k values for $z^{1/k}$ (the kth root of z) since

$$z^{1/k} = [re^{j(\theta+2\pi n)}]^{1/k} = r^{1/k}e^{j(\theta+2\pi n)/k}, \quad n = 0, 1, \ldots, k-1$$

- the value for $k = 0$ is the *principal value* of $z^{1/n}$

Logarithms of complex numbers: taking log of $z = re^{j\theta} = re^{j(\theta \pm 2\pi n)}$, $n = 0, 1, 2, \ldots$, we have

$$\ln z = \ln r + j(\theta \pm 2\pi n), \quad n = 0, 1, 2, \ldots$$

- $\ln z$ for $n = 0$ is the *principal value* of $\ln z$ and is denoted by $\mathrm{Ln}\ z$
- properties of logarithms hold for complex arguments

$$\log(z_1 z_2) = \log z_1 + \log z_2, \quad \log(z_1/z_2) = \log z_1 - \log z_2$$

$$a^{(z_1+z_2)} = a^{z_1} \times a^{z_2}, \quad z^c = e^{c \ln z}, \quad a^z = e^{z \ln a}$$

Examples: for $n = 0, 1, 2, \ldots$, we have

$$\ln 1 = \ln(1e^{\pm j2\pi n}) = \pm j2\pi n, \qquad \ln(-1) = \ln(1e^{\pm j\pi(2n+1)}) = \pm j\pi(2n+1)$$

$$\ln j = \ln(e^{j\pi(1\pm 4n)/2}) = j\pi\frac{1\pm 4n}{2}, \qquad j^j = e^{j\ln j} = e^{-\pi(1\pm 4n)/2}$$

Outline

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- **sinusoids and exponentials**
- vectors and matrices
- matrix calculus
- partial fraction expansion

Sinusoid

$$x(t) = A\cos(2\pi f_0 t + \theta)$$

- A is the *amplitude*
- θ is the *phase* (in degrees of radians)
- f_0 is the *frequency* (in Hertz)
- since $\cos(\phi) = \cos(\phi + 2\pi n)$ for any integer n, the angle $2\pi f_0 t + \theta$ changes by 2π when t changes by $1/f_0$; hence there are f_0 repetitions per second
- $T_0 = 1/f_0$ is the *period*, which is the repetition interval
- the *radian frequency* is $\omega_0 = 2\pi f_0 = 2\pi/T_0$

Sinusoids and phasors

the **phasor** of the sinusoid $A\cos(\omega t + \theta)$ is the complex number $Ae^{j\theta} = A\angle\theta$

Adding sinusoids

- two sinusoids having the *same* frequency can be added using trigonometric identities or using *phasors*

$$A_1\cos(\omega t + \theta_1) + A_2\cos(\omega t + \theta_2) = A\cos(\omega t + \theta)$$

- A and θ can computed by using phasors:

$$A_1 e^{j\theta_1} + A_2 e^{j\theta_2} = Ae^{j\theta}$$

Example: find $\cos(\omega t + 60^o) + 5\cos(\omega t - 30^o)$

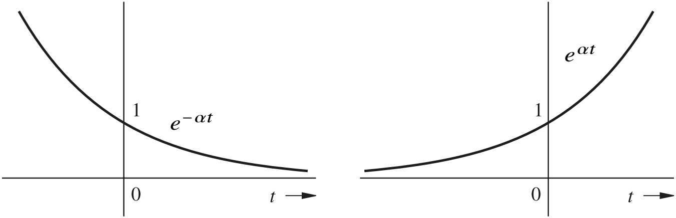
- we have $e^{j60^o} + 5e^{-j30^o} = 5.099e^{-j18.69^o} = Ae^{j\theta}$
- therefore,

$$\cos(\omega t + 60^o) + 5\cos(\omega t - 30^o) = 5.099\cos(\omega t - 18.69^o)$$

Exponentials

the *exponential function* is $e^{\alpha t}$

- for $\alpha > 0$, $e^{-\alpha t}$ decays monotonically, and $e^{\alpha t}$ grows monotonically with t



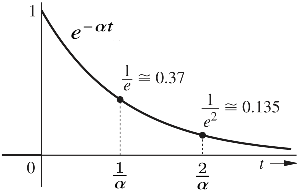
(a) (b)

- exponentials and sinusoids are related as

$$\cos\phi = \frac{1}{2}(e^{j\phi} + e^{-j\phi}), \qquad \sin\phi = \frac{1}{2j}(e^{j\phi} - e^{-j\phi})$$

Sketching exponentials

- $e^{-\alpha t} = 1$ at $t = 0$ and at $t = 1/\alpha$, the value drops to $1/e$ (37% of its initial value)
- the time interval over which the exponential reduces by factor of e is called *time constant*; thus, time constant of $e^{-\alpha t}$ is $\tau = 1/\alpha$
- $e^{-\alpha t}$ is reduced to 37% of its initial value over any time interval of duration $\frac{1}{\alpha}$



(a) (b)

- monotonically growing exponentials, the waveform increases by a factor e over each interval of $1/\alpha$ seconds

Exponentially varying sinusoid

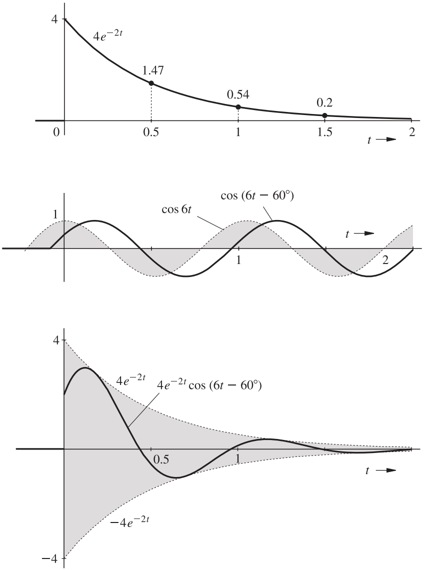
an *exponentially varying sinusoid*

$$x(t) = Ae^{-\alpha t}\cos(\omega_0 t + \theta)$$

can be sketched by

1. sketching $Ae^{-\alpha t}$
2. sketching $-Ae^{-\alpha t}$
3. constraining the amplitude of $\cos(\omega_0 t + \theta)$

Example: $4e^{-2t}\cos(6t - 60^o)$



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Vector

an n *column vector* is an ordered list of n numbers, represented by:

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

- the ith *entry* (or element, coefficient, component) of vector $\boldsymbol{a}$ is denoted by a_i
- the number of entries it contains, n, is *size* or *dimension*
- we also use $\boldsymbol{a} = (a_1, \ldots, a_n)$ to denote an n-column vector
- *transpose* of an n-column vector $\boldsymbol{a}$ is the *row vector*

$$\boldsymbol{a}^{\mathsf{T}} = (a_1, \ldots, a_n)^{\mathsf{T}} = [a_1 \; a_2 \; \cdots \; a_n]$$

Block (partitioned) vectors

vectors can be stacked (concatenated, partitioned) to create larger vectors

Example: if $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are vectors of size n, m, p, then $\boldsymbol{d} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is the $(m + n + p)$-vector:

$$\boldsymbol{d} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} = (a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_p).$$

If $\boldsymbol{a} = (1, 2)$, $\boldsymbol{b} = (5, 9)$, and $\boldsymbol{d} = (-1, 3)$, then

$$\boldsymbol{d} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = (1, 2, 5, 9, -1, 3)$$

Special vectors

One and zero vectors

$$\mathbf{1} = (1, 1, \ldots, 1), \quad \mathbf{0} = (0, 0, \ldots, 0)$$

(size follow from context or we write $\mathbf{1}_n$, $\mathbf{0}_n$)

Unit vectors

- for any integer k, the *unit vectors* are $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_k$
- $\boldsymbol{e}_i$ is a vector with zero entries except entry $e_i = 1$
- for $k = 3$, we have

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Vector addition

given two n-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ of equal size, we have:

$$\boldsymbol{a}+\boldsymbol{b}=\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}, \qquad \boldsymbol{a}-\boldsymbol{b}=\begin{bmatrix} a_1-b_1 \\ a_2-b_2 \\ \vdots \\ a_n-b_n \end{bmatrix}.$$

Properties

- *commutative:* $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$
- *associative:* $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$

Vector addition

- the vector $\boldsymbol{a} - \boldsymbol{b}$ is called the difference between $\boldsymbol{a}$ and $\boldsymbol{b}$
- two vectors $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^n$ are equal if $\boldsymbol{a} - \boldsymbol{b} = \mathbf{0}$, *i.e.*,

$$a_i = b_i \quad \text{for all } i = 1, 2, \ldots, n$$

- the vector $\mathbf{0} - \boldsymbol{a}$ is denoted by $-\boldsymbol{a}$
- the vector $\boldsymbol{x}$ that solves the equation

$$\boldsymbol{a} + \boldsymbol{x} = \boldsymbol{b}$$

  is $\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{a}$

Scalar-vector multiplication

for vector $\boldsymbol{a} \in \mathbb{R}^n$ and scalar α:

$$\alpha \boldsymbol{a} = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$$

Properties

- *distributive:* for any real scalars α and β,

$$\alpha(\boldsymbol{a} + \boldsymbol{b}) = \alpha \boldsymbol{a} + \alpha \boldsymbol{b}$$
$$(\alpha + \beta)\boldsymbol{a} = \alpha \boldsymbol{a} + \beta \boldsymbol{a}$$

- *associative:* $\alpha(\beta \boldsymbol{a}) = (\alpha\beta)\boldsymbol{a}$; as a convention, we write $\alpha\beta\boldsymbol{a} = \alpha(\beta\boldsymbol{a}) = (\alpha\beta)\boldsymbol{a}$

Matrix

an $m \times n$ **matrix** is a rectangular array of numbers, written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- a_{ij} is the i, j entry (element) located at ith row and jth column
- size or dimension is $m \times n$ (#rows $\times$ # columns)
- *transpose* of A is the $n \times m$ matrix A^T with entries $a^T_{ij} = a_{ji}$; for example

$$\begin{bmatrix} 1 & 4 & 0 \\ 2 & 6 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 4 & 6 \\ 0 & 3 \end{bmatrix}$$

- a matrix A is square if $m = n$ ($n \times n$ matrix); a square matrix is **symmetric** $A = A^T$ ($a_{ij} = a_{ji}$)

Special matrices

a **zero matrix** is a matrix with all zero elements, denoted by 0

- the size of the zero matrix is determined from the context
- the zero matrix of size $m \times n$ is sometimes written as $0_{m\times n}$

a **diagonal matrix** is square matrix (size $n \times n$) whose elements are zero everywhere except on the main diagonal; for example

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \triangleq \mathrm{diag}(2, 1, 5)$$

the **identity matrix** of size n, denoted by $\boldsymbol{I}$ is the diagonal matrix with unity for all its diagonal elements

- size determined from context or written as $\boldsymbol{I}_n$
- examples"

$$\boldsymbol{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the 2×2 and 3×3 identity matrices

Block (partitioned) matrices

Matrices can be represented in term of submatrices

Example: is 2×2 *block* matrix

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

- entries B, C, D, and E are called *blocks* or *submatrices*
- the submatrices can be referred to by their block row and column indices; for example, C is the $(1,2)$ block of A
- block matrices must have compatible dimensions
- if

$$B = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}, \quad C = [-1], \quad D = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 5 \end{bmatrix}, \quad E = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

  then

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & -1 \\ 2 & 2 & 1 & 4 \\ 1 & 3 & 5 & 4 \end{bmatrix}$$

Columns and rows of a matrix

a matrix can be viewed as a block matrix with row/column vector blocks

- $m \times n$ matrix $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}$$

  where $\boldsymbol{a}_j$ denotes the jth column $\boldsymbol{a}_j = (a_{1j}, \ldots, a_{mj})$, for $j = 1, \ldots, n$

- $m \times n$ matrix $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \begin{bmatrix} \bar{\boldsymbol{a}}_1^{\mathsf{T}} \\ \bar{\boldsymbol{a}}_2^{\mathsf{T}} \\ \vdots \\ \bar{\boldsymbol{a}}_m^{\mathsf{T}} \end{bmatrix},$$

  where $\bar{\boldsymbol{a}}_i^{\mathsf{T}}$ is the ith rows defined as $\bar{\boldsymbol{a}}_i^{\mathsf{T}} = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix}$ for $i = 1, \ldots, m$

Matrix addition

two matrices $\boldsymbol{A}, \boldsymbol{B}$ of the same size ($m \times n$) can be added together element wise

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

Properties.

- *commutativity:* $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
- *associativity:* $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$. We thus write both as $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}$
- *transpose of sum:* $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$

Scalar matrix multiplication

for matrix A and scalar α, we have

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}$$

Example

$$(-3) \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 6 \end{bmatrix}$$

Properties.

- $(\alpha A)^T = \alpha A^T$ for any scalar α
- for scalars α and β, it holds that

$$(\alpha + \beta)A = \alpha A + \beta A, \quad (\alpha\beta)A = \alpha(\beta A)$$

Matrix-matrix multiplication

for $m \times n$ matrix A and $n \times p$ matrix B, then

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

is the $m \times p$ matrix with entries:

$$c_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

- size of A and B must be compatible (conformable):

  # columns in $\boldsymbol{A}$ = # rows in $\boldsymbol{B}$

- the order of matrix multiplication is not commutative in general
  - $\boldsymbol{AB}$ is not always the same as $\boldsymbol{BA}$
  - if $\boldsymbol{A}$ is an $m \times p$ matrix and $\boldsymbol{B}$ is an $p \times n$ matrix, then $\boldsymbol{BA}$ does not make sense if $m \neq n$

Examples

- $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 12 & 18 \end{bmatrix}$$

- for an $m \times n$ matrix, we have

$$\boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A}$$

Properties of matrix multiplication

(for scalar α and matrices $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$)

- *associativity*

$$(\boldsymbol{A}\boldsymbol{B})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{C}),$$

  which we write it as $\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}$.

- *associativity with scalar multiplication*

$$\alpha(\boldsymbol{A}\boldsymbol{B}) = (\alpha\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(\alpha\boldsymbol{B})$$

  We thus write it as $\alpha\boldsymbol{A}\boldsymbol{B}$

- *distributivity with addition*

$$\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\boldsymbol{B}+\boldsymbol{A}\boldsymbol{C} \text{ and } (\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{A}\boldsymbol{C}+\boldsymbol{B}\boldsymbol{C}$$

- *transpose of product*

$$(\boldsymbol{A}\boldsymbol{B})^{\mathsf{T}} = \boldsymbol{B}^{\mathsf{T}}\boldsymbol{A}^{\mathsf{T}}$$

Matrix-vector multiplication

for $m \times n$ matrix $\boldsymbol{A}$ and n-vector $\boldsymbol{x}$, we have

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \qquad y_i = \sum_{j=1}^{n} a_{ij} x_j$$

- $\boldsymbol{y}$ is an m-vector
- example:

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (0)(1) + (1)(2) + (2)(3) \\ (-1)(1) + (0)(2) + (-1)(3) \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

Properties.

1. *distributive:* $\boldsymbol{A}(\boldsymbol{u} + \mathrm{v}) = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{A}\boldsymbol{v}$ and $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u}$ where $\boldsymbol{u}, \boldsymbol{v}$ are vectors and $\boldsymbol{A}, \boldsymbol{B}$ are matrices
2. $(\alpha \boldsymbol{A})\boldsymbol{u} = \alpha(\boldsymbol{A}\boldsymbol{u}) = \boldsymbol{A}(\alpha \boldsymbol{u})$; as convention, we write it as $\alpha \boldsymbol{A}\boldsymbol{u}$

Matrix determinant

ijth submatrix of $\boldsymbol{A}$: if $\boldsymbol{A}$ is an $n \times n$ matrix, then the ijth *submatrix* of $\boldsymbol{A}$, denoted by $\boldsymbol{A}_{ij}$, is the $(m-1) \times (m-1)$ obtained by deleting row i and column j of $\boldsymbol{A}$; for example,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \qquad \boldsymbol{A}_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad \boldsymbol{A}_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

Determinant: the *determinant* of a matrix is computed a follows; pick any value of i ($i = 1, 2, \ldots, n$) and compute

$$|\boldsymbol{A}| = \sum_{j=1}^{n} (-1)^{i+j} |\boldsymbol{A}_{ij}| a_{ij},$$

- the quantities $|\boldsymbol{A}_{ij}|$ and $(-1)^{i+j}|\boldsymbol{A}_{ij}|$ are called the *minor* and *cofactor* of element a_{ij}
- for $n \times n$ matrices $\boldsymbol{A}, \boldsymbol{B}$, $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$

Example

a) for a scalar matrix $\boldsymbol{A} = [a_{11}]$, we have $|\boldsymbol{A}| = a_{11}$

b) for a 2×2 matrix, the determinant is

$$|\boldsymbol{A}| = \left|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right| = a_{11}a_{22} - a_{21}a_{12}$$

c) for the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

– we have for $i = 1$

$$\boldsymbol{A}_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \quad \boldsymbol{A}_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad \boldsymbol{A}_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

– thus, the determinant is

$$\begin{aligned} |\boldsymbol{A}| &= (-1)^2 a_{11}|\boldsymbol{A}_{11}| + (-1)^3 a_{12}|\boldsymbol{A}_{12}| + (-1)^4 a_{13}|\boldsymbol{A}_{13}| \\ &= a_{11}|\boldsymbol{A}_{11}| - a_{12}|\boldsymbol{A}_{12}| + a_{13}|\boldsymbol{A}_{13}| \\ &= 1(-3) - 2(-6) + 3(-3) = 0 \end{aligned}$$

Properties of determinants

Multiplication of a single row/column by a constant: if a single row or column of a matrix, $\boldsymbol{A}$, is multiplied by a constant, k, forming the matrix, $\tilde{\boldsymbol{A}}$, then

$$\det \tilde{\boldsymbol{A}} = k \det \boldsymbol{A}$$

Multiplication of all elements by a constant

$$\det(k\boldsymbol{A}) = k^n \det \boldsymbol{A}$$

Transpose

$$\det \boldsymbol{A}^T = \det \boldsymbol{A}$$

Determinant of the product of square matrices

$$\det \boldsymbol{A}\boldsymbol{B} = \det \boldsymbol{A} \det \boldsymbol{B}$$

$$\det \boldsymbol{A}\boldsymbol{B} = \det \boldsymbol{B}\boldsymbol{A}$$

Inverse

the matrix $\boldsymbol{A}^{-1}$ is said to be the **inverse** of the $n \times n$ matrix $\boldsymbol{A}$ if it satisfies

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = I_n$$

- invertible matrices must be *square*
- if $\boldsymbol{A}$ has an inverse $\boldsymbol{A}^{-1}$, then the inverse of $\boldsymbol{A}^{-1}$ is $\boldsymbol{A}$
- for a non-zero scalar a, the inverse is the number x such that $ax = 1$, which we denote by $x = 1/a = a^{-1}$
- if the inverse of A exists, then the matrix is said to be *invertible* or *nonsingular*
- a *square matrix* $\boldsymbol{A}$ is invertible if and only if the determinant is nonzero ($|\boldsymbol{A}| \neq 0$)

Example

a) the identity matrix $\boldsymbol{I}$ is invertible, with inverse $\boldsymbol{I}^{-1} = \boldsymbol{I}$ since

$$(\boldsymbol{I})\boldsymbol{I} = \boldsymbol{I}$$

b) any 2×2 matrix $\boldsymbol{A}$ is invertible if and only if $a_{11}a_{22} \neq a_{12}a_{21}$, with inverse

$$\boldsymbol{A}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

c) a diagonal matrix

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

is invertible if and only if $d_{ii} \neq 0$ for $i = 1, \ldots, n$, and

$$D^{-1} = \begin{bmatrix} 1/d_{11} & 0 & \cdots & 0 \\ 0 & 1/d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_{nn} \end{bmatrix}$$

Inverse properties

- **Inverse of transpose:** if A is invertible, its transpose A^T is also invertible with inverse:

$$(A^T)^{-1} = (A^{-1})^T$$

- **Inverse of matrix product:** if both $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible square matrices of the same size, then $\boldsymbol{AB}$ is invertible, and

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

- **Negative matrix power:** let $\boldsymbol{A}$ be a square invertible matrix, then

$$(\boldsymbol{A}^p)^{-1} = (\boldsymbol{A}^{-1})^p$$

for any integer p

Square linear equation

set or system of n linear equations with n variables $x_1, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

- scalars a_{ij} are called coefficients and the numbers b_i are called right-hand-sides.
- using matrix notation:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},$$

  where the $n \times n$ matrix A is called the coefficient matrix and the m vector $\boldsymbol{b}$ is called the right-hand side

Cramers's rule

if the determinant $|\boldsymbol{A}| \neq 0$, then the square linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution given by *Cramer's formula*

$$x_k = \frac{|\boldsymbol{D}_k|}{|\boldsymbol{A}|}, \quad k = 1, 2, \ldots, n$$

- $\boldsymbol{D}_k$ is the matrix obtained replacing the kth column of $\boldsymbol{A}$ by the right-hand side (column) $\boldsymbol{b}$
- by definition, we know that

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

  it follows from Cramer's formula (with some algebra) that

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|} \underbrace{\begin{bmatrix} |\boldsymbol{A}_{11}| & |\boldsymbol{A}_{21}| & \cdots & |\boldsymbol{A}_{n1}| \\ |\boldsymbol{A}_{12}| & |\boldsymbol{A}_{22}| & \cdots & |\boldsymbol{A}_{n1}| \\ \vdots & \vdots & \ddots & \vdots \\ |\boldsymbol{A}_{1n}| & |\boldsymbol{A}_{2n}| & \cdots & |\boldsymbol{A}_{nn}| \end{bmatrix}}_{\text{adj } \boldsymbol{A}}$$

Rank of a matrix

the *rank* of a matrix, $\boldsymbol{A}$, equals the number of linearly independent rows or columns

rank can be found by finding the highest-order square submatrix that is nonsingular; for example,

$$\boldsymbol{A} = \begin{bmatrix} 1 & -5 & 2 \\ 4 & 7 & -5 \\ -3 & 15 & -6 \end{bmatrix}$$

since the determinant is zero, the 3×3 matrix is singular; choosing the submatrix

$$\boldsymbol{A}_{33} = \begin{bmatrix} 1 & -5 \\ 4 & 7 \end{bmatrix}$$

whose determinant equals 27, we conclude that $\operatorname{rank} \boldsymbol{A} = 2$

Eigenvalues and eigenvectors

for an $(n \times n)$ square matrix $\mathbf{A}$, any vector $\mathbf{x}$ ($\mathbf{x} \neq 0$) that satisfies the equation

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

is an *eigenvector*, and λ is the corresponding *eigenvalue*

- eigenvalues are solution of the *characteristic equation*

$$Q(\lambda) = |\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 \lambda^0 = 0$$

- the polynomial $Q(\lambda)$ is called the *characteristic polynomial* of matrix $\mathbf{A}$

Cayley-Hamilton theorem: every $n \times n$ matrix $\mathbf{A}$ satisfies its own characteristic equation

$$\mathbf{Q}(\mathbf{A}) = \mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1 \mathbf{A} + a_0 \mathbf{A}^0 = 0$$

Outline

- complex numbers
- sinusoids and exponentials
- vectors and matrices
- **matrix calculus**
- partial fraction expansion

Derivative and integral of matrix

$$\mathbf{A}(t) = \left[a_{ij}(t)\right]_{m\times n}$$

the derivative and integral of a A with respect to t are defined entrywise:

$$\frac{d}{dt}[\mathbf{A}(t)] = \left[\frac{d}{dt}a_{ij}(t)\right]_{m\times n} \quad \text{or} \quad \dot{\mathbf{A}}(t) = \left[\dot{a}_{ij}(t)\right]_{m\times n}$$

$$\int \mathbf{A}(t)dt = \left[\int a_{ij}(t)dt\right]_{m\times n}$$

Example:

$$\mathbf{A}(t) = \begin{bmatrix} e^{-2t} & \sin t \\ e^{t} & e^{-t} + e^{-2t} \end{bmatrix}$$

the derivative of $A(t)$ is

$$\dot{\mathbf{A}}(t) = \begin{bmatrix} -2e^{-2t} & \cos t \\ e^{t} & -e^{-t} - 2e^{-2t} \end{bmatrix}$$

the integral of $A(t)$ is

$$\int \mathbf{A}(t) dt = \begin{bmatrix} \int e^{-2t} dt & \int \sin dt \\ \int e^{t} dt & \int \left(e^{-t} + 2e^{-2t}\right) dt \end{bmatrix}$$

Derivative properties

Linearity

- $\dfrac{d}{dt}(\mathbf{A}+\mathbf{B}) = \dfrac{d\mathbf{A}}{dt} + \dfrac{d\mathbf{B}}{dt}$
- $\dfrac{d}{dt}(c\mathbf{A}) = c\dfrac{d\mathbf{A}}{dt}$

Matrix product

- $\dfrac{d}{dt}(\mathbf{A}\mathbf{B}) = \dfrac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\dfrac{d\mathbf{B}}{dt} = \dot{\mathbf{A}}\mathbf{B} + \mathbf{A}\dot{\mathbf{B}}$
- If we let $\mathbf{B} = \mathbf{A}^{-1}$, we obtain

$$\frac{d}{dt}\left(\mathbf{A}\mathbf{A}^{-1}\right) = \frac{d\mathbf{A}}{dt}\mathbf{A}^{-1} + \mathbf{A}\frac{d}{dt}\mathbf{A}^{-1} = 0$$

hence

$$\frac{d}{dt}\left(\mathbf{A}^{-1}\right) = -\mathbf{A}^{-1}\frac{d\mathbf{A}}{dt}\mathbf{A}^{-1}$$

Functions of a matrix

consider the function:

$$f(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda_2^2 + \cdots = \sum_{i=0}^{\infty} \alpha_i \lambda^i$$

if λ is an eigenvalue of $\mathbf{A}$, then from characteristic equation, we have

$$\lambda^n = -a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \cdots - a_1\lambda - a_0$$

hence λ^{n+k} can be expressed in terms of $\lambda^{n-1}, \lambda^{n-2}, \ldots, \lambda$ for any k; therefore,

$$f(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_{n-1}\lambda^{n-1}$$

for some β_i and

$$\begin{bmatrix} f(\lambda_1) \\ f(\lambda_2) \\ \vdots \\ f(\lambda_n) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}$$

if we assume that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct, then:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} f(\lambda_1) \\ f(\lambda_2) \\ \vdots \\ f(\lambda_n) \end{bmatrix}$$

now if $f(\mathbf{A})$ is a function of a square matrix $\mathbf{A}$:

$$f(\mathbf{A}) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A} + \alpha_2 \mathbf{A}^2 + \cdots = \sum_{i=0}^{\infty} \alpha_i \mathbf{A}^i$$

then using Cayley-Hamilton theorem, we can show that

$$f(\mathbf{A}) = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} + \beta_2 \mathbf{A}^2 + \cdots + \beta_{n-1} \mathbf{A}^{n-1} = \sum_{i=0}^{n-1} \beta_i \mathbf{A}^i$$

where the coefficients β_i are found as before (if some of the eigenvalues are repeated (multiple roots), the results should be modified)

Exponential of a matrix

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^n t^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}$$

we have

$$e^{\mathbf{A}t} = \sum_{i=1}^{n-1} \beta_i \mathbf{A}^i$$

where

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

Example 1

compute $e^{\mathbf{A}t}$ for the case $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Solution: the characteristic equation is

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

hence, the eigenvalues are $\lambda_1 = -1, \lambda_2 = -2$, and

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

therefore,

$$\begin{aligned} e^{\mathbf{A}t} &= \beta_0 \mathbf{I} + \beta_1 \mathbf{A} = \left(2e^{-t} - e^{-2t}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(e^{-t} - e^{-2t}\right) \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \end{aligned}$$

Matrix power

we can express $\mathbf{A}^k$ as

$$\mathbf{A}^k = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} + \cdots + \beta_{n-1} \mathbf{A}^{n-1}$$

where

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1^k \\ \lambda_2^k \\ \vdots \\ \lambda_n^k \end{bmatrix}$$

Outline

- complex numbers
- sinusoids and exponentials
- vectors and matrices
- matrix calculus
- **partial fraction expansion**

Rational functions

a rational function $F(s)$ can be expressed as

$$F(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0} = \frac{N(s)}{D(s)}$$

- a and b are real constants, and m and n are positive integers
- the function $F(s)$ is proper if $m < n$ and improper if $m \geq n$
- only a proper rational function can be expanded as a sum of partial fractions
- for an improper function $F(s)$, we can always separate it into a sum of a polynomial in s and a proper function

Improper functions

Example: consider the improper function

$$F(s) = \frac{2s^3 + 9s^2 + 11s + 2}{s^2 + 4s + 3}$$

we can divide the numerator by the denominator:

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad 2s + 1 \\ \hline s^2 + 4s + 3 & \; 2s^3 + 9s^2 + 11s + 2 \\ & -2s^3 - 8s^2 - 6s \\ \hline & \quad\quad\; s^2 + 5s + 2 \\ & \quad\; -s^2 - 4s - 3 \\ \hline & \quad\quad\quad\quad\quad s - 1 \end{array}$$

therefore, $F(s)$ can be expressed as

$$F(s) = \frac{2s^3 + 9s^2 + 11s + 2}{s^2 + 4s + 3} = \underbrace{2s + 1}_{\text{polynomial in } s} + \underbrace{\frac{s - 1}{s^2 + 4s + 3}}$$

Partial fraction expansion

we can factor the denominator of $F(s)$ and express it as

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-\lambda_1)(s-\lambda_2)\ldots(s-\lambda_n)}$$

- $\lambda_1, \ldots, \lambda_n$ are the roots of the characteristic equations $D(s) = 0$
- for each multiple root of $D(s)$ of multiplicity r, the expansion contains r terms

Example:

$$\frac{s+6}{s(s+3)(s+1)^2},$$

the denominator has four roots; two distinct at $s = 0$ and $s = -3$ and multiple root of multiplicity 2 occurs at $s = -1$; thus the partial fraction expansion of this function takes the form

$$\frac{s+6}{s(s+3)(s+1)^2} = \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}.$$

Example: method of clearing fractions

obtain a partial fraction expansion of

$$F(s) = \frac{s^3 + 3s^2 + 4s + 6}{(s+1)(s+2)(s+3)^2}$$

Solution: $F(s)$ can be expressed as a sum of partial fractions:

$$F(s) = \frac{s^3 + 3s^2 + 4s + 6}{(s+1)(s+2)(s+3)^2} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3} + \frac{K_4}{(s+3)^2}$$

to find the constants K_i, we clear fractions by multiplying both sides by $(s+1)(s+2)(s+3)^2$:

$$\begin{aligned} s^3 + 3s^2 + 4s + 6 &= K_1\left(s^3 + 8s^2 + 21s + 18\right) + K_2\left(s^3 + 7s^2 + 15s + 9\right) \\ &\quad + K_3\left(s^3 + 6s^2 + 11s + 6\right) + K_4\left(s^2 + 3s + 2\right) \\ &= s^3\left(K_1 + K_2 + K_3\right) + s^2\left(8K_1 + 7K_2 + 6K_3 + K_4\right) \\ &\quad + x\left(21K_1 + 15K_2 + 11K_3 + 3K_4\right) + \left(18K_1 + 9K_2 + 6K_3 + 2K_4\right) \end{aligned}$$

equating coefficients of similar powers on both sides yields

$$\begin{aligned} K_1 + K_2 + K_3 &= 1 \\ 8K_1 + 7K_2 + 6K_3 + K_4 &= 3 \\ 21K_1 + 15K_2 + 11K_3 + 3K_4 &= 4 \\ 18K_1 + 9K_2 + 6K_3 + 2K_4 &= 6 \end{aligned}$$

solving these four equations gives

$$K_1 = 1, \quad K_2 = -2, \quad K_3 = 2, \quad K_4 = -3$$

hence,

$$F(s) = \frac{1}{s+1} - \frac{2}{s+2} + \frac{2}{s+3} - \frac{3}{(s+3)^2}$$

- this method is straightforward but cumbersome
- we next develop easier methods

The Method of residues: distinct factors

Distinct factors: suppose $F(s) = N(s)/D(s)$ ($m < n$)

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)}$$
$$= \frac{K_1}{s-\lambda_1} + \frac{K_2}{s-\lambda_2} + \cdots + \frac{K_n}{s-\lambda_n}$$

where λ_i distinct

Method of residues: we can determine the coefficient K_j by:

$$K_j = (s-\lambda_j)F(s)\Big|_{s=\lambda_j} \quad j = 1, 2, \ldots, n$$

this procedure is also called the *Heaviside "cover-up" method*

Example: distinct real factors

$$F(s) = \frac{2s^2 + 9s - 11}{(s+1)(s-2)(s+3)}$$
$$= \frac{K_1}{s+1} + \frac{K_2}{s-2} + \frac{K_3}{s+3}$$

to find K_1 corresponding to the factor $(s+1)$, we cancel (cover-up) the term $(s+1)$ in the denominator of $F(s)$ and then substitute $s = -1$:

$$K_1 = \left.\frac{2s^2 + 9s - 11}{\cancel{(s+1)}(s-2)(s+3)}\right|_{s=-1} = \frac{2 - 9 - 11}{(-1-2)(-1+3)} = \frac{-18}{-6} = 3$$

similarly, to compute K_2, we cover up the factor $(s-2)$ in $F(s)$ and let $s=2$ in the remaining function, as follows:

$$K_2 = \left. \frac{2s^2+9s-11}{(s+1)\cancel{(s-2)}(s+3)} \right|_{s=2} = \frac{8+18-11}{(2+1)(2+3)} = \frac{15}{15} = 1$$

and

$$K_3 = \left. \frac{2s^2+9s-11}{(s+1)(s-2)\cancel{(s+3)}} \right|_{s=-3} = \frac{18-27-11}{(-3+1)(-3-2)} = \frac{-20}{10} = -2$$

therefore,

$$F(s) = \frac{2s^2+9s-11}{(s+1)(s-2)(s+3)} = \frac{3}{s+1} + \frac{1}{s-2} - \frac{2}{s+3}$$

Example: distinct complex factors

complex factors conjugate can be treated the same as distinct factors

$$F(s) = \frac{4s^2 + 2s + 18}{(s+1)(s+2-j3)(s+2+j3)} = \frac{K_1}{s+1} + \frac{K_2}{s+2-j3} + \frac{K_3}{s+2+j3}$$

we have

$$K_1 = \left.\frac{4s^2 + 2s + 18}{\cancel{(s+1)}\left(s^2+4s+13\right)}\right|_{s=-1} = 2$$

$$K_2 = \left.\frac{4s^2 + 2s + 18}{(s+1)\cancel{(s+2-j3)}(s+2+j3)}\right|_{s=-2+j3} = 1 + j2 = \sqrt{5}e^{j63.43^\circ}$$

$$K_3 = \left.\frac{4s^2 + 2s + 18}{(s+1)(s+2-j3)\cancel{(s+2+j3)}}\right|_{s=-2-j3} = 1 - j2 = \sqrt{5}e^{-j63.43^\circ}$$

therefore

$$F(s) = \frac{2}{s+1} + \frac{\sqrt{5}e^{j63.43^\circ}}{s+2-j3} + \frac{\sqrt{5}e^{-j63.43^\circ}}{s+2+j3}$$

- if the coefficients of a rational function are real, then coefficients corresponding to the complex-conjugate factors are conjugates of each other
- in such a case, we need to compute only one of the coefficients

Quadratic factors

it is more convenient to combine the two terms arising from complex-conjugate factors into one quadratic factor

Example: $F(s)$ the previous example can be expressed as

$$F(s) = \frac{4s^2 + 2s + 18}{(s+1)(s^2 + 4s + 13)} = \frac{2}{s+1} + \frac{As + B}{s^2 + 4s + 13}$$

the values of A and B can be determined by clearing fractions:

$$\begin{aligned} 4s^2 + 2s + 18 &= 2\left(s^2 + 4s + 13\right) + (As + B)(s+1) \\ &= (2 + A)\, s^2 + (8 + A + B)\, s + (26 + B) \end{aligned}$$

equating terms of similar powers yields $A = 2$, $B = -8$; hence

$$F(s) = \frac{4s^2 + 2s + 18}{(s+1)\,(s^2 + 4s + 13)} = \frac{2}{s+1} + \frac{2s - 8}{s^2 + 4s + 13}$$

Finding quadratic constant using substitution approach: we can also find the quadratic constants A and B by substituting convenient values of s in both sides

Example:

$$F(s) = \frac{4s^2 + 2s + 18}{(s+1)(s^2 + 4s + 13)} = \frac{2}{s+1} + \frac{As + B}{s^2 + 4s + 13}$$

plugging in $s = 0$ on both sides gives

$$\frac{18}{13} = 2 + \frac{B}{13} \quad \Rightarrow \quad B = -8$$

to find A, we can multiply both sides of

$$\frac{4s^2 + 2s + 18}{(s+1)\,(s^2 + 4s + 13)} = \frac{2}{s+1} + \frac{As + B}{s^2 + 4s + 13}$$

by s and then let $s \longrightarrow \infty$ (when $s \longrightarrow \infty$, only the terms of the highest power are significant); therefore,

$$4 = 2 + A \quad \Rightarrow \quad A = 2$$

Example: depending on the function, we can use other convenient values for s:

$$F(s) = \frac{2s^2 + 4s + 5}{x\,(s^2 + 2s + 5)} = \frac{1}{s} + \frac{As + B}{s^2 + 2s + 5}$$

if we plug in $s = 0$, we obtain ∞ on both sides! it is more convenient to use $s = 1$:

$$\frac{11}{8} = 1 + \frac{A + B}{8} \quad \text{or} \quad A + B = 3$$

we can now choose some other value for s (*e.g.*, $s = 2$) to obtain one more equation to solve for A and B; in this case, a simple method is to multiply both sides by s and then let $s \to \infty$ to get

$$2 = 1 + A \quad \Rightarrow \quad A = 1$$

hence, $B = 3 - c_1 = 2$ and

$$F(s) = \frac{1}{s} + \frac{s + 2}{s^2 + 2s + 5}$$

Repeated factors

if the root $\lambda_n = \hat{\lambda}$ of $D(s) = 0$, is repeated r times, then

$$\begin{aligned} F(s) &= \frac{N(s)}{(s-\hat{\lambda})^r (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_{n-r})} \\ &= \frac{\hat{K}_1}{(s-\hat{\lambda})} + \frac{\hat{K}_2}{(s-\hat{\lambda})^2} + \cdots + \frac{\hat{K}_r}{(s-\hat{\lambda})^r} \\ &\quad + \frac{K_1}{s-\lambda_1} + \frac{K_2}{s-\lambda_2} + \cdots + \frac{K_{n-r}}{s-\lambda_{n-r}} \end{aligned}$$

- coefficients $K_1, K_2, \ldots, K_{n-r}$ corresponding to the unrepeated factors can be found using any of the previous methods
- coefficients $\hat{K}_1, K_2, \ldots, \hat{K}_r$ can be obtained by

$$\hat{K}_\ell = \frac{1}{(r-\ell)!} \frac{d^{r-\ell}}{ds^{r-\ell}} \left[(s-\lambda)^r F(s)\right]\Big|_{s=\hat{\lambda}}, \quad \ell = 1, \ldots, r$$

to illustrate, assume $n = r = 4$, then

$$F(s) = \frac{\hat{K}_1}{(s-\hat{\lambda})} + \frac{\hat{K}_2}{(s-\hat{\lambda})^2} + \frac{\hat{K}_3}{(s-\hat{\lambda})^3} + \frac{\hat{K}_4}{(s-\hat{\lambda})^4}$$

and

$$\begin{aligned} \hat{K}_4 &= \left[(s-\lambda)^r F(s)\right]\big|_{s=\hat{\lambda}} \\ \hat{K}_3 &= \frac{d}{ds}\left[(s-\lambda)^r F(s)\right]\bigg|_{s=\hat{\lambda}} \\ \hat{K}_2 &= \frac{1}{2!}\frac{d^2}{ds^2}\left[(s-\lambda)^r F(s)\right]\bigg|_{s=\hat{\lambda}} \\ \hat{K}_1 &= \frac{1}{3!}\frac{d^3}{ds^3}\left[(s-\lambda)^r F(s)\right]\bigg|_{s=\hat{\lambda}} \end{aligned}$$

Example

$$F(s) = \frac{4s^3 + 16s^2 + 23s + 13}{(s+1)^3(s+2)}$$

$$= \frac{\hat{K}_1}{s+1} + \frac{\hat{K}_2}{(s+1)^2} + \frac{\hat{K}_3}{(s+1)^3} + \frac{k}{s+2}$$

the coefficient k corresponds to unrepeated facor $\lambda = -2$, so

$$k = \left. \frac{4s^3 + 16s^2 + 23s + 13}{(s+1)^3\cancel{(s+2)}} \right|_{s=-2} = 1$$

to find $\hat{K}_3$, we conceal the factor $(s+1)^3$ in $F(s)$ and let $s = -1$:

$$\hat{K}_3 = \left. \frac{4s^3 + 16s^2 + 23s + 13}{\cancel{(s+1)^3}(s+2)} \right|_{s=-1} = 2$$

to find $\hat{K}_2$, we conceal the factor $(s+1)^3$ in $F(s)$, take the derivative of the remaining expression, and then let $s = -1$:

$$\hat{K}_2 = \frac{d}{ds}\left[\frac{4s^3+16s^2+23s+13}{\cancel{(s+1)^3}(s+2)}\right]\Bigg|_{s=-1} = 1$$

similarly,

$$\hat{K}_1 = \frac{1}{2!}\frac{d^2}{ds^2}\left[\frac{4s^3+16s^2+23s+13}{\cancel{(s+1)^3}(s+2)}\right]\Bigg|_{s=-1} = 3$$

therefore,

$$F(s) = \frac{3}{s+1} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3} + \frac{1}{s+2}$$

Combination of residue method and shortcuts

consider the previous example where

$$F(s) = \frac{4s^3 + 16s^2 + 23s + 13}{(s+1)^3(s+2)}$$
$$= \frac{\hat{K}_1}{s+1} + \frac{\hat{K}_2}{(s+1)^2} + \frac{2}{(s+1)^3} + \frac{1}{s+2}$$

to avoid taking derivatives, we can multiply both sides of this equation by s and then let $s \to \infty$, we can eliminate $\hat{K}_2$:

$$4 = \hat{K}_1 + 1 \quad \Longrightarrow \quad \hat{K}_1 = 3$$

thus,

$$\frac{4s^3 + 16s^2 + 23s + 13}{(s+1)^3(s+2)} = \frac{3}{s+1} + \frac{\hat{K}_2}{(s+1)^2} + \frac{2}{(s+1)^3} + \frac{1}{s+2}$$

plugging $s = 0$, we have

$$\frac{13}{2} = 2 + \hat{K}_2 + 3 + \frac{1}{2} \Rightarrow \hat{K}_2 = 1$$

Improper $F(s)$ with $m = n$

when the numerator and denominator polynomials of $F(s)$ have the same degree $(m = n)$:

$$F(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$
$$= a_n + \frac{K_1}{s - \lambda_1} + \frac{K_2}{s - \lambda_2} + \cdots + \frac{K_n}{s - \lambda_n}$$

- the coefficients $K_1, K_2, \ldots, K_n$ are computed as if $F(s)$ were proper
- the only difference between the proper and improper case is the appearance of an extra constant a_n in the latter

Example: expand $F(s)$ into partial fractions if

$$F(s) = \frac{3s^2 + 9s - 20}{s^2 + s - 6} = \frac{3s^2 + 9s - 20}{(s-2)(s+3)}$$

here, $m = n = 2$ with $a_n = a_2 = 3$; thus,

$$F(s) = \frac{3s^2 + 9s - 20}{(s-2)(s+3)} = 3 + \frac{K_1}{s-2} + \frac{K_2}{s+3}$$

in which

$$K_1 = \left.\frac{3s^2 + 9s - 20}{(s-2)(s+3)}\right|_{s=2} = \frac{12 + 18 - 20}{(2+3)} = \frac{10}{5} = 2$$

and

$$K_2 = \left.\frac{3s^2 + 9s - 20}{(s-2)(s+3)}\right|_{s=-3} = \frac{27 - 27 - 20}{(-3-2)} = \frac{-20}{-5} = 4$$

hence,

$$F(s) = \frac{3s^2 + 9s - 20}{(s-2)(s+3)} = 3 + \frac{2}{s-2} + \frac{4}{s+3}$$

General improper rational functions

for an improper function $F(s)$, we can always separate it into a sum of a polynomial in s and a proper function:

$$F(s) = \sum_{\ell=0}^{m-n} \alpha_\ell s^\ell + \frac{N_1(s)}{D(s)}$$

Example:

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

dividing the denominator into the numerator until the remainder is a proper rational function gives

$$F(s) = s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20},$$

where the term $(30s + 100)/\left(s^2 + 9s + 20\right)$ is the remainder we expand the proper rational function into a sum of partial fractions:

$$\frac{30s + 100}{s^2 + 9s + 20} = \frac{30s + 100}{(s + 4)(s + 5)} = \frac{-20}{s + 4} + \frac{50}{s + 5}$$

hence

$$F(s) = s^2 + 4s + 10 - \frac{20}{s + 4} + \frac{50}{s + 5}$$

MATLAB

the partial fraction expansion of rational function $F(s) = B(s)/A(s)$ can be computed using the `residue` command in MATALB:

```
[R,P,K] = residue(B,A)
```

- two input vectors `B` and `A` specify the polynomial coefficients of the numerator and denominator, respectively; these vectors are ordered in descending powers of the independent variable
- output vector `R` contains the coefficients of each partial fraction, and vector `P` contains the corresponding roots of each partial fraction. For a root repeated r times, the r partial fractions are ordered in ascending powers
- when the rational function is not proper, the vector `K` contains the direct terms, which are ordered in descending powers of the independent variable

Example: let us use MATLAB to find the partial fraction expansion of

$$F(s) = \frac{s^5 + \pi}{s^4 - \sqrt{8}s^3 + \sqrt{32}s - 4}$$

MATLAB code:

```
>> [R,P,K] = residue([1 0 0 0 0 pi],[1 -sqrt(8) 0 sqrt(32) -4]);

>> R.', P.', K
R = 7.8888 5.9713 3.1107 0.1112
P = 1.4142 1.4142 1.4142 -1.4142
K = 1.0000 2.8284
```

hence

$$F(s) = s + 2.8284 + \frac{7.8888}{s - \sqrt{2}} + \frac{5.9713}{(s - \sqrt{2})^2} + \frac{3.1107}{(s - \sqrt{2})^3} + \frac{0.1112}{s + \sqrt{2}}$$