9. The z-transform

- the (unilateral) z-transform
- inverse z-transform
- properties of z-transform

Definition

the z-*transform* of the sequence $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- z is a complex variable
- $x[n]$ is called the *inverse* z-transform of $X(z)$ (notation: $x[n] \Longleftrightarrow X_(z)$)
- the *region of convergence* (ROC) for $X(z)$ are the values of z (in the complex plane) for which the sum converges (exists)

Linearity: if

$$x_1[n] \Longleftrightarrow X_1(z) \quad \text{and} \quad x_2[n] \Longleftrightarrow X_2(z)$$

then

$$a_1x_1[n] + a_2x_2[n] \Longleftrightarrow a_1X_1(z) + a_2X_2(z)$$

Example

the z-transform for the signal $\gamma^n u[n]$ is

$$X(z) = \sum_{n=\infty}^{\infty} \gamma^n u[n] z^{-n} = \sum_{n=0}^{\infty} (\gamma/z)^n = \frac{1}{1-\frac{\gamma}{z}} = \frac{z}{z-\gamma} \qquad |\gamma/z| < 1 \ (|z| > |\gamma|)$$

for $|z| < |\gamma|$, the sum does not converge (it goes to infinity)

the z-transform of $-\gamma^n u[-(n+1)]$ is

$$X(z) = \sum_{n=\infty}^{\infty} -\gamma^n u[-(n+1)] z^{-n} = \sum_{n=-1}^{-\infty} -(\gamma/z)^n = z/(z-\gamma)$$

with *different* ROC $|z| < |\gamma|$

Restriction to causal signals

- so the inverse z-transform of $X(z)$ is not unique
- restricting the $x[n]$ to be causal, then the inverse transform is unique
- for unilateral transform, we ignore the ROC in determining the inverse z-transform

The unilateral z-transform

the *unilateral z-transform* is defined for causal signals:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Existence: the existence of the z-transform is guaranteed if

$$|X(z)| \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z|^n} < \infty, \quad \text{for some } |z|$$

- if $|x[n]| \leq r_0^n$ for some r_0, then

$$|X(z)| \leq \sum_{n=0}^{\infty} (r_0/|z|)^n = \frac{1}{1 - \frac{r_0}{|z|}}, \quad |z| > r_0$$

  and hence, $X(z)$ exists for $|z| > r_0$

- some signal models, *e.g.*, γ^{n^2}, are not z-transformable

Example 9.1

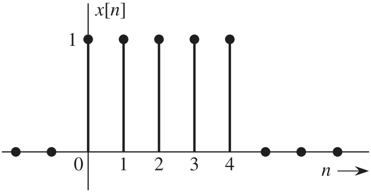
using the definition, find the z-transform of:

(a) $\delta[n]$

(b) $u[n]$

(c) $\cos(\beta n)u[n]$

(d) the signal $x[n]$ shown below



Solution:

(a)

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \delta[n]z^{-n} = 1, \quad \text{for all } z$$

(b) for $x[n] = u[n]$, we have

$$\begin{aligned} X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} &= \sum_{n=0}^{\infty} z^{-n} \\ &= \frac{1}{1-\frac{1}{z}} \qquad 1/z < 1 \\ &= \frac{z}{z-1} \qquad |z| > 1 \end{aligned}$$

(c) using $\cos(\beta n) = \left(e^{j\beta n} + e^{-j\beta n}\right)/2$ and

$$e^{\pm j\beta n}u[n] \Longleftrightarrow \frac{z}{z - e^{\pm j\beta}} \quad |z| > \left|e^{\pm j\beta}\right| = 1$$

we have

$$X(z) = \frac{1}{2}\left[\frac{z}{z - e^{j\beta}} + \frac{z}{z - e^{-j\beta}}\right] = \frac{z(z - \cos\beta)}{z^2 - 2z\cos\beta + 1} \quad |z| > 1$$

(d) here, we have

$$X(z) = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} = \frac{z^4 + z^3 + z^2 + z + 1}{z^4} \quad \text{for all } z \neq 0$$

or

$$X(z) = \sum_{n=0}^{4} z^{-n} = \frac{\left(\frac{1}{z}\right)^5 - \left(\frac{1}{z}\right)^0}{\frac{1}{z} - 1} = \frac{z}{z - 1}\left(1 - z^{-5}\right)$$

Outline

- the (unilateral) z-transform
- **inverse z-transform**
- properties of z-transform

Inverse z-transform

the *inverse z-transform* of $X(z)$ can be computed as follows:

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

- $\oint$ is integration in counterclockwise direction around closed path in complex plane
- we won't use this integration as we use transform table to find inverse z-transform

Notation

$$X(z) = \mathcal{Z}\{x[n]\} \quad \text{and} \quad x[n] = \mathcal{Z}^{-1}\{X(z)\}$$

$$x[n] \Longleftrightarrow X(z)$$

note that

$$\mathcal{Z}^{-1}[\mathcal{Z}\{x[n]\}] = x[n] \quad \text{and} \quad \mathcal{Z}\left[\mathcal{Z}^{-1}\{X(z)\}\right] = X(z)$$

Inverse transform using known z-transform pairs

many of the transforms $X(z)$ of practical interest are rational functions

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_{M-1} z + b_M}{(z - p_1)(z - p_2) \ldots (z - p_N)}$$

where $P(z)$ and $Q(z)$ are polynomials

- $X(z)$ is called *proper* if $M < N$ and *improper* if $M \geq N$
- *zeros* of $X(z)$ are values of z for which $X(z) = 0$ (*e.g.*, $P(z) = 0$)
- *poles* of $X(z)$ are values of z for which $X(z) \to \infty$ (*e.g.*, $Q(z) = 0$)
- we can obtain $x[n]$ from known pairs given the partial-fraction expansion of $X(z)$, which expresses $X(z)$ as a sum of fractions with simpler denominator
- the ROC is the region of z-plane to outside of all the finite poles of $X(z)$

Example 9.2

find the inverse z-transforms of

$$\frac{8z - 19}{(z-2)(z-3)}$$

Solution: expanding $X(z)$ into partial fractions yields

$$X(z) = \frac{8z - 19}{(z-2)(z-3)} = \frac{3}{z-2} + \frac{5}{z-3}$$

from table (pair 5), we obtain

$$x[n] = \left[3(2)^{n-1} + 5(3)^{n-1}\right] u[n-1]$$

Modified partial fraction expansion

to obtain a form that contains $u[n]$ instead of $u[n-1]$, we expand:

$$\frac{X(z)}{z} = \text{partial expansion} \Rightarrow x[n] = \mathcal{Z}^{-1}\{z \times \text{partial expansion}\}$$

Example: from the last example,

$$\frac{X(z)}{z} = \frac{8z-19}{z(z-2)(z-3)} = \frac{(-19/6)}{z} + \frac{(3/2)}{z-2} + \frac{(5/3)}{z-3}$$

multiplying both sides by z yields

$$X(z) = -\frac{19}{6} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$$

using pairs 1 and 4 in table, we get

$$x[n] = -\frac{19}{6}\delta[n] + \left[\frac{3}{2}(2)^n + \frac{5}{3}(3)^n\right]u[n]$$

Example 9.3

find the inverse z-transforms of

(a) $\dfrac{z\left(2z^2 - 11z + 12\right)}{(z-1)(z-2)^3}$

(b) $\dfrac{2z(3z+17)}{(z-1)\left(z^2 - 6z + 25\right)}$

Solution:

(a) we have repeated poles and we expand as

$$\frac{X(z)}{z} = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

where

$$k = \left.\frac{2z^2 - 11z + 12}{(z-1)(z-2)^3}\right|_{z=1} = -3 \qquad a_0 = \left.\frac{2z^2 - 11z + 12}{(z-1)(z-2)^3}\right|_{z=2} = -2$$

therefore,

$$\frac{X(z)}{z} = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

to find a_2, we multiply both sides by z and let $z \to \infty$:

$$0 = -3 - 0 + 0 + a_2 \Longrightarrow a_2 = 3$$

letting z take any convenient value, say, $z = 0$, on both sides:

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2}$$

which yields $a_1 = -1$; therefore,

$$\frac{X(z)}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{z-2}$$

and

$$X(z) = -3\frac{z}{z-1} - 2\frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3\frac{z}{z-2}$$

using pairs 4, 6, and 9 of table, gives

$$\begin{aligned} x[n] &= \left[-3 - 2\frac{n(n-1)}{8}(2)^n - \frac{n}{2}(2)^n + 3(2)^n\right] u[n] \\ &= -[3 + \tfrac{1}{4}\left(n^2 + n - 12\right) 2^n]u[n] \end{aligned}$$

(b) we have complex poles:

$$\frac{X(z)}{z} = \frac{2(3z+17)}{(z-1)\,(z^2-6z+25)} = \frac{2(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

we find the partial fraction of $X(z)/z$ using the Heaviside "cover-up" method:

$$\frac{X(z)}{z} = \frac{2}{z-1} + \frac{1.6e^{-j2.246}}{z-3-j4} + \frac{1.6e^{j2.246}}{z-3+j4}$$

and

$$X(z) = 2\frac{z}{z-1} + \left(1.6e^{-j2.246}\right)\frac{z}{z-3-j4} + \left(1.6e^{j2.246}\right)\frac{z}{z-3+j4}$$

using pair 12b (table) with $r/2 = 1.6$, $\theta = -2.246\ \mathrm{rad}$, $\gamma = 3+j4 = 5e^{j0.927}$, so that $|\gamma| = 5, \beta = 0.927$; therefore, we have

$$x[n] = [2 + 3.2(5)^n \cos(0.927n - 2.246)]\,u[n]$$

method of quadratic factors: we can instead expand as

$$\frac{X(z)}{z} = \frac{2(3z+17)}{(z-1)\,(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

multiplying both sides by z and letting $z \to \infty$, we find

$$0 = 2 + A \Longrightarrow A = -2$$

and

$$\frac{2(3z+17)}{(z-1)\,(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

to find B, we let z take any convenient value, say, $z = 0$:

$$\frac{-34}{25} = -2 + \frac{B}{25} \Longrightarrow B = 16$$

therefore,

$$\frac{X(z)}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25}$$

and

$$X(z) = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

we now use pair 12c (table) with $A = -2$, $B = 16$, $|\gamma| = 5$, and $a = -3$, so that

$$r = \sqrt{\frac{100+256-192}{25-9}} = 3.2, \quad \beta = \cos^{-1}\left(\frac{3}{5}\right) = 0.927\,\mathrm{rad}$$

and

$$\theta = \tan^{-1}\left(\frac{-10}{-8}\right) = -2.246\,\mathrm{rad}$$

hence

$$x[n] = [2 + 3.2(5)^n \cos(0.927n - 2.246)]\, u[n]$$

Inverse transform by power series expansion

by definition,

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0]z^0 + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots$$

- if we can expand $X(z)$ into the power series in z^{-1}, the coefficients of this power series can be identified as $x[0], x[1], x[2], x[3], \ldots$
- a rational $X(z)$ can be expanded into a power series of z^{-1} by dividing its numerator by the denominator
- this is useful if we want to know only the first few terms of $x[n]$

Example

$$X(z) = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} = \frac{7z^3 - 2z^2}{z^3 - 1.7z^2 + 0.8z - 0.1}$$

we have:

$$X(z) = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} = 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots$$

therefore,

$$x[0] = 7, \quad x[1] = 9.9, \quad x[2] = 11.23, \quad x[3] = 11.87, \quad \ldots$$

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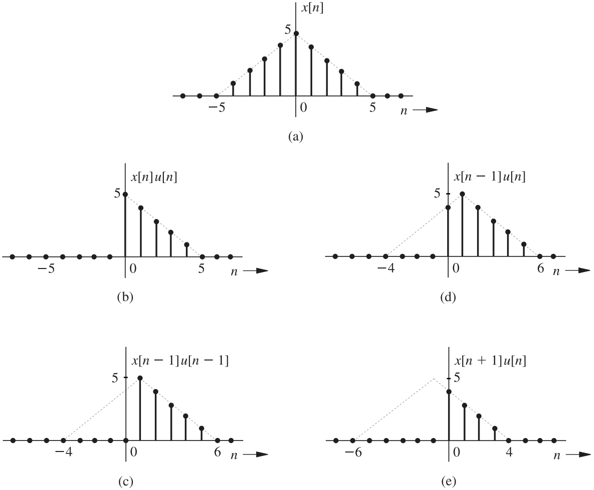
Shifting forms

shifting forms: $x[n]u[n]$, $x[n-m]u[n]$, $x[n-m]u[n-m]$, and $x[n+m]u[n]$



Shifting properties

Causal-part right-shift: if $x[n]u[n] \iff X(z)$ then for integer value of m,

$$x[n-m]u[n-m] \iff \frac{1}{z^m}X(z)$$

Right-shift: integer value of $m > 0$,

$$x[n-m]u[n] \iff z^{-m}X(z) + z^{-m}\sum_{n=1}^{m} x[-n]z^n$$

for $m = 1$:

$$x[n-1]u[n] \iff \frac{1}{z}X(z) + x[-1]$$

Left-shift: for integer value of $m > 0$,

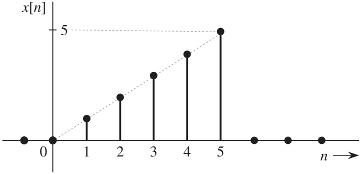
$$x[n+m]u[n] \iff z^m X(z) - z^m \sum_{n=0}^{m-1} x[n]z^{-n}$$

for $m = 1$:

$$x[n+1]u[n] \iff zX(z) - zx[0]$$

Example 9.4

use the shifting property and the z-transform table to find the z-transform of $x[n]$



Solution: $x[n]$ can be expressed as:

$$x[n] = n(u[n] - u[n-6]) = nu[n] - nu[n-6]$$

to use the right-shift property, we rearrange $nu[n-6]$ in terms of $(n-6)u[n-6]$:

$$\begin{aligned} x[n] &= nu[n] - (n-6+6)u[n-6] \\ &= nu[n] - (n-6)u[n-6] - 6u[n-6] \end{aligned}$$

because $u[n] \iff z/(z-1)$,

$$u[n-6] \iff \frac{1}{z^6}\frac{z}{z-1} = \frac{1}{z^5(z-1)}$$

also, because $nu[n] \iff z/(z-1)^2$

$$(n-6)u[n-6] \iff \frac{1}{z^6}\frac{z}{(z-1)^2} = \frac{1}{z^5(z-1)^2}$$

therefore,

$$X(z) = \frac{z}{(z-1)^2} - \frac{1}{z^5(z-1)^2} - \frac{6}{z^5(z-1)} = \frac{z^6-6z+5}{z^5(z-1)^2}$$

Time and frequency reversal

Time-reversal: if $x[n] \iff X(z)$, then

$$x[-n] \iff X(1/z)$$

if ROC of $x[n]$ is $|z| > |\gamma|$, then ROC of $x[-n]$ is $|z| < 1/|\gamma|$

Example: find the z-transform and ROC of $u[-n]$

from table $u[n] \iff U(z) = z/(z-1)$ with ROC $|z| > 1$, hence,

$$u[-n] \iff U(1/z) = \frac{1/z}{(1/z)-1} = \frac{1}{z-1} \quad \text{with inverted ROC } |z| < 1$$

Frequency-reversal: if $x[n] \iff X(z)$, then

$$(-1)^n x[n] \iff X(-z)$$

Scaling and differentiation in the z-domain

Scaling in z-domain (multiplication by γ^n)

if $x[n]u[n] \Longleftrightarrow X(z)$, then

$$\gamma^n x[n]u[n] \Longleftrightarrow X(z/\gamma)$$

Differentiation in z-domain (multiplication by n)

if $x[n]u[n] \Longleftrightarrow X(z)$, then

$$nx[n]u[n] \Longleftrightarrow -z\frac{d}{dz}X(z)$$

Example 9.5

find the unilateral z-transform of $x[n] = (1-n)\cos(\frac{\pi}{2}(n-1))u[n-1]$

Solution: using properties:

$$\cos(\pi n/2)u[n] \Longleftrightarrow \frac{z^2}{z^2+1} \quad \text{(table pair 10)}$$

$$-n\cos(\pi n/2)u[n] \Longleftrightarrow z\frac{d}{dz}\left(\frac{z^2}{z^2+1}\right) \quad (z\text{-domain differentiation})$$

$$= z\left(\frac{2z}{z^2+1} - \frac{z^2}{(z^2+1)^2}(2z)\right) = \frac{2z^2}{(z^2+1)^2}$$

$$x[n] = -(n-1)\cos(\pi(n-1)/2)u[n-1] \Longleftrightarrow z^{-1}\frac{2z^2}{(z^2+1)^2} \quad \text{(time shift)}$$

therefore,

$$X(z) = \frac{2z}{(z^2+1)^2} = \frac{2z}{z^4+2z^2+1}$$

Time accumulation

if $x[n] \Longleftrightarrow X(z)$ then

$$\sum_{k=0}^{n} x[k] = \frac{z}{z-1} X(z)$$

Example: we know that $x[n] = \delta[n]$ has z-transform $X[z] = 1$; we can use the fact

$$u[n] = \sum_{k=0}^{n} \delta[n] = \sum_{k=0}^{n} x[n]$$

and accumulation property to show that

$$u[n] = \sum_{k=0}^{n} x[n] \Longleftrightarrow \frac{z}{z-1}(1) = \frac{z}{z-1}$$

Convolution

Time-convolution: if

$$x_1[n] \Longleftrightarrow X_1(z) \quad \text{and} \quad x_2[n] \Longleftrightarrow X_2(z)$$

then

$$x_1[n] * x_2[n] \Longleftrightarrow X_1(z)X_2(z)$$

Example: using z-transform compute $u[n] * u[n-1]$

from convolution and shifting properties, we have

$$u[n] * u[n-1] \iff \frac{z}{z-1}\frac{z}{z-1}\frac{1}{z} = \frac{z}{(z-1)^2}$$

using the table, we have

$$u[n] * u[n-1] = nu[n] \iff \frac{z}{(z-1)^2}$$

Initial and final values

Initial value theorem: for a causal $x[n]$, we have

$$x[0] = \lim_{z \to \infty} X(z)$$

Final value theorem: if $(z-1)X(z)$ has no poles outside the unit circle, then

$$\lim_{N \to \infty} x[N] = \lim_{z \to 1} (z-1)X(z)$$

References

Reference:

- B. P. Lathi, R. A. Green. *Linear Systems and Signals*. Oxford University Press, 2018.