8. Analysis using Laplace transform

- the transfer function
- stability
- frequency response
- LTI systems realization
- introduction to feedback system design*

Transfer function

the *transfer function* of LTIC system is the Laplace transform of its impulse response:

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

- $h(t)$ is the impulse response (output due to impulse input $\delta(t)$)
- the response of an LTIC system to an exponential $x(t) = e^{st}$ is

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = H(s) e^{st}$$

- for LTI system with input $x(t) = e^{st}$, output is of the same form $y(t) = H(s)e^{st}$
  - such input is called *eigenfunction*
- an alternate definition of the transfer function $H(s)$ of an LTI system , is

$$H(s) = \left. \frac{\text{output signal}}{\text{input signal}} \right|_{\text{input}=e^{st}}$$

Zero-state response

taking Laplace transform of $y(t) = x(t) * h(t)$, we have

$$Y(s) = X(s)H(s)$$

- $H(s)$ is called transfer function because it describes in the s domain how the system "transfers" the excitation to the response
- if we know $H(s)$ and $X(s)$, then

$$y(t) = \mathcal{L}^{-1}[X(s)H(s)]$$

Transfer function of LTI differential system

$$Q(D)y(t) = P(D)x(t)$$

or

$$\begin{aligned}&(D^N + a_1 D^{N-1} + \cdots + a_{N-1}D + a_N)y(t)\\ &= (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1}D + b_N)x(t)\end{aligned}$$

- the transfer function for this system is

$$H(s) = \frac{P(s)}{Q(s)}$$

- for an LTI differential system, the transfer function is simple to obtain

Example 8.1

consider an LTIC system described by the equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

find the transfer function and the zero-state response if the input $x(t) = 3e^{-5t}u(t)$

Solution: the system equation is

$$\underbrace{\left(D^2+5D+6\right)}_{Q(D)}y(t)=\underbrace{(D+1)}_{P(D)}x(t)$$

therefore,

$$H(s)=\frac{P(s)}{Q(s)}=\frac{s+1}{s^2+5s+6}$$

since

$$X(s)=\mathcal{L}\left[3e^{-5t}u(t)\right]=\frac{3}{s+5}$$

we have

$$Y(s)=X(s)H(s)=\frac{3(s+1)}{(s+5)\left(s^2+5s+6\right)}=\frac{-2}{s+5}-\frac{1}{s+2}+\frac{3}{s+3}$$

the inverse Laplace transform of this equation is

$$y(t)=\left(-2e^{-5t}-e^{-2t}+3e^{-3t}\right)u(t)$$

Example 8.2

show that the transfer function of:

(a) an ideal delay of T seconds is e^{-sT}

(b) an ideal differentiator is s

(c) an ideal integrator is $1/s$

Solution:

(a) for an ideal delay of T seconds, the input $x(t)$ and output $y(t)$ are related by

$$y(t) = x(t - T) \quad \text{and} \quad Y(s) = X(s)e^{-sT}$$

therefore,

$$H(s) = \frac{Y(s)}{X(s)} = e^{-sT}$$

(b) for an ideal differentiator, the input $x(t)$ and the output $y(t)$ are related by

$$y(t) = \frac{dx(t)}{dt}$$

the Laplace transform of this equation is

$$Y(s) = sX(s) \quad [x(0^-) = 0 \text{ for a causal signal}]$$

hence

$$H(s) = \frac{Y(s)}{X(s)} = s$$

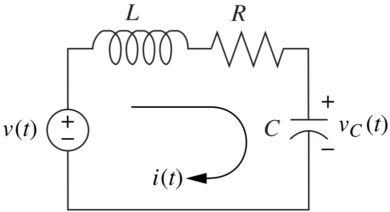
(c) for an ideal integrator with zero initial state, $y(0^-) = 0$,

$$y(t) = \int_0^t x(\tau)d\tau \quad \text{and} \quad Y(s) = \frac{1}{s}X(s)$$

therefore,

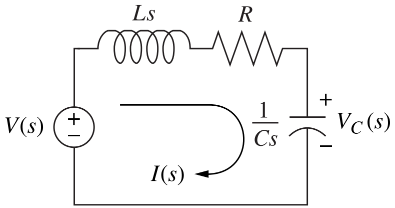
$$H(s) = \frac{1}{s}$$

Example 8.3



find the transfer function relating $V_C(s)$ to input voltage $V(s)$

Solution: the Laplace circuit is



the voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances; thus,

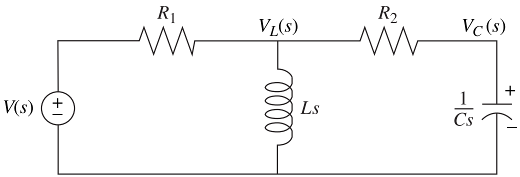
$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)} V(s)$$

solving for the transfer function, $V_C(s)/V(s)$, yields

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Example 8.4

find the transfer function, $V_C(s)/V(s)$, for the circuit using nodal analysis



recall that the admittance, $Y(s)$ is the reciprocal of impedance:

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$$

when writing nodal equations, it can be more convenient to use admittance

Solution: the sum of currents flowing from the nodes marked $V_L(s)$ and $V_C(s)$ are

$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0$$

$$CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0$$

rearranging and using conductances, $G_1 = 1/R_1$ and $G_2 = 1/R_2$, we obtain,

$$\left(G_1 + G_2 + \frac{1}{Ls}\right) V_L(s) \quad - G_2 V_C(s) = V(s) G_1$$

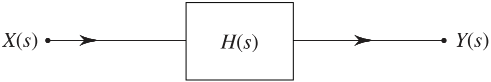
$$-G_2 V_L(s) + (G_2 + Cs)\, V_C(s) = 0$$

solving for the transfer function, $V_C(s)/V(s)$, yields

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1 G_2}{C} s}{(G_1 + G_2)\, s^2 + \frac{G_1 G_2 L + C}{LC} s + \frac{G_2}{LC}}$$

Block diagrams

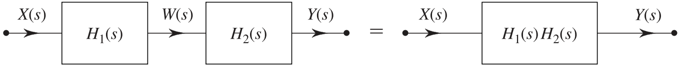
we can represent an LTI system using its transfer function using block diagrams



- $Y(s) = X(s)H(s)$
- large systems are conveniently represented by block diagrams

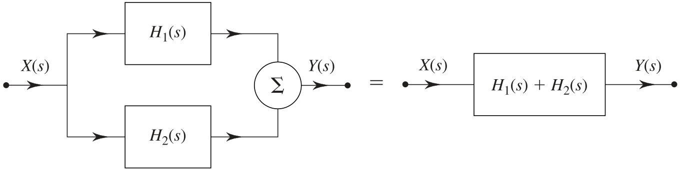
Cascade and parallel connections

Cascade interconnection

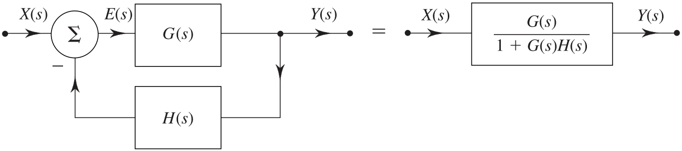


$$\frac{Y(s)}{X(s)} = \frac{W(s)}{X(s)}\frac{Y(s)}{W(s)} = H_1(s)H_2(s)$$

Parallel interconnection



Feedback interconnection



$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Inverse systems

if $H(s)$ is the t.f. of a system $\mathcal{S}$, then the t.f. of its inverse system $\mathcal{S}_i$ is

$$H_i(s) = \frac{1}{H(s)}$$

- this follows from the fact that $h(t) * h_i(t) = \delta(t)$, implying $H(s)H_i(s) = 1$
- for example, an ideal integrator and its inverse, an ideal differentiator, have transfer functions $1/s$ and s, respectively, leading to $H(s)H_i(s) = 1$

Outline

- the transfer function
- **stability**
- frequency response
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BIBO Stability

given transfer function $H(s) = P(s)/Q(s)$, then the LTI system is

- BIBO-stable if the poles of $H(s)$ are in LHP (excluding $j\omega$-axis)
- BIBO-unstable if at least one pole of $H(s)$ is *not* in LHP

Improper system: if $M > N$, then the system is BIBO-unstable

- this is because, using long division, we obtain $H(s) = R(s) + H_p(s)$, where $R(s)$ is an $(M - N)$th-order polynomial and $H_p(s)$ is a proper transfer function
- for example,

$$H(s) = \frac{s^3 + 4s^2 + 4s + 5}{s^2 + 3s + 2} = s + \frac{s^2 + 2s + 5}{s^2 + 3s + 2}$$

  the term s is the transfer function of an ideal differentiator

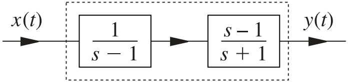
- applying unit-step function, the output will contain an impulse (unbounded output)

Asymptotic (internal) stability

if $H(s) = \dfrac{P(s)}{Q(s)}$ and $P(s)$, $Q(s)$ have **no common factors**, then the LTI system is

1. *asymptotically stable* if and only if all the poles of $H(s)$ are in the LHP
2. *marginally stable* if and only if there are no poles of $H(s)$ in the RHP and some unrepeated poles on the imaginary axis
3. *unstable* if and only if either one or both of the following conditions exist:
   (i) at least one pole of $H(s)$ is in the RHP;
   (ii) there are repeated poles of $H(s)$ on the imaginary axis

Example 8.5



determine the BIBO and asymptotic stability of the composite (cascade) system

Solution: the transfer function of the cascade system is

$$H(s) = H_1(s)H_2(s) = \left(\frac{1}{s-1}\right)\left(\frac{s-1}{s+1}\right) = \frac{1}{s+1}$$

thus, the system is BIBO-stable since all poles are in LHP

to determine the asymptotic stability,

- note that $\mathcal{S}_1$ has one characteristic root at 1, and $\mathcal{S}_2$ also has one root at -1
- hence, the composite system has a RHP root at $+1$, and is asymptotically unstable

Example 8.6

consider a feedback system on page 8.15 with

$$G(s) = K/s(s+8) \quad \text{and} \quad H(s) = 1$$

determine the transfer function and BIBO stability of the system when:
(a) $K = 7$; (b) $K = 16$; (c) $K = 80$

Solution: we have

$$H_{\text{feedback}}(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K/(s(s+8))}{1 + K/(s(s+8))} = \frac{K}{s^2 + 8s + K}$$

hence

(a) $H_{\text{feedback}}(s) = 7/(s^2 + 8s + 7)$, the poles are $s = -1, -7$ on LHP, hence stable

(b) $H_{\text{feedback}}(s) = 7/(s^2 + 8s + 16)$, the poles are $s = -4, -4$ on LHP, hence stable

(c) $H_{\text{feedback}}(s) = 7/(s^2 + 8s + 80)$, the poles are $s = -4 \pm 8j$ on LHP, hence stable

Matlab feedback function

we can use MATLAB `feedback` function to determine the transfer function in the previous example

(a)
```
>> H = tf(1,1); K = 7; G = tf([0 0 K],[1 8 0]);
TFa = feedback(G,H)
Ha =
7
-------------
s^2 + 8 s + 7
```

(b)
```
>> H = tf(1,1); K = 16; G = tf([0 0 K],[1 8 0]);
TFb = feedback(G,H)
Hb =
16
--------------
s^2 + 8 s + 16
```

(c)
```
>> H = tf(1,1); K = 80; G = tf([0 0 K],[1 8 0]);
TFc = feedback(G,H)
Hc =
80
--------------
s^2 + 8 s + 80
```

Outline

- the transfer function
- stability
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Frequency response

Frequency response: the response of an LTI system $h(t)$ to complex sinusoid

$$x(t) = A_x e^{j\omega t} = |A_x| e^{j(\omega t + \angle A_x)}$$

is

$$\begin{aligned} y(t) = \int_{-\infty}^{\infty} h(\tau) A_x e^{j\omega(t-\tau)} d\tau &= H(j\omega) A_x e^{j\omega t} \\ &= |H(j\omega)||A_x| e^{j(\omega t + \angle A_x + \angle H(j\omega))} \end{aligned}$$

- $H(j\omega)$ is called the *frequency response* of the system
- the *amplitude* of the output is $|H(j\omega)|$ times the input amplitude
- the *phase* of the output is shifted by $\angle H(j\omega)$ with respect to the input phase
- frequency response allows us determine the system output to any sinusoidal input

Sinusoidal input: for input $\cos(\omega t + \theta) = \mathrm{Re}(e^{j(\omega t + \theta)})$, system response is:

$$y(t) = |H(j\omega)| \cos[\omega t + \theta + \angle H(j\omega)]$$

Amplitude and phase responses

Amplitude response

- $|H(j\omega)|$ is the amplitude gain called *amplitude response* or *magnitude response*
- plot $|H(j\omega)|$ v ω shows the amplitude gain as a function of frequency ω

Phase response

- $\angle H(j\omega)$ is the *phase response*
- plot $\angle H(j\omega)$ v ω shows how the system changes the phase of input sinusoid

Example 8.7

an LTIC system is described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3000\frac{dy(t)}{dt} + 2 \times 10^6 y(t) = 2 \times 10^6 x(t)$$

(a) find its transfer function

(b) find $y(t)$ if $x(t) = 3e^{j\pi/2}e^{j400\pi t}$

(c) find $y(t)$ if $x(t) = 8\cos(200\pi t)$

Solution:

(a) the transfer function is

$$H(s) = \frac{2 \times 10^6}{s^2 + 3000s + 2 \times 10^6}$$

(b) the frequency response is

$$H(j\omega) = \frac{2 \times 10^6}{(j\omega)^2 + 3000(j\omega) + 2 \times 10^6} = \frac{2 \times 10^6}{2 \times 10^6 - \omega^2 + j3000\omega}$$

using $\omega = 400\pi$, we have $H(j400\pi) = 0.5272e^{-j1.46}$, hence

$$y(t) = (|H(j400\pi)| \times 3)e^{j(\angle H(j400\pi) + \pi/2)}e^{j400\pi t} = 1.582e^{j(400\pi t + 0.1112)}$$

(c) we have

$$\begin{aligned} y(t) &= |H(j200\pi)| \times 8\cos(200\pi t + \angle H(j200\pi)) \\ &= 0.8078 \times 8\cos(200\pi t - 0.8654) = 6.4625\cos(200\pi t - 0.8654) \end{aligned}$$

Example 8.8

find the frequency response of a system whose transfer function is

$$H(s) = \frac{s + 0.1}{s + 5}$$

also, find the system response $y(t)$ if the input $x(t)$ is

(a) $\cos 2t$

(b) $\cos(10t - 50°)$

Solution:

$$H(j\omega) = \frac{j\omega + 0.1}{j\omega + 5}$$

therefore,

$$|H(j\omega)| = \frac{\sqrt{\omega^2 + 0.01}}{\sqrt{\omega^2 + 25}} \quad \text{and} \quad \angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$$

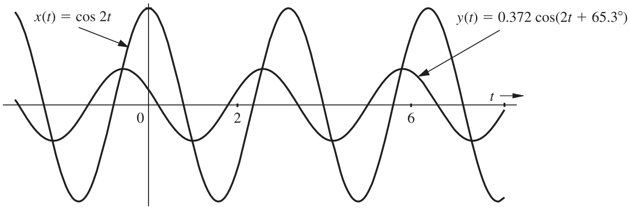
(a) for the input $x(t) = \cos 2t, \omega = 2$, and

$$|H(j2)| = \frac{\sqrt{(2)^2 + 0.01}}{\sqrt{(2)^2 + 25}} = 0.372$$

$$\angle H(j2) = \tan^{-1}\left(\frac{2}{0.1}\right) - \tan^{-1}\left(\frac{2}{5}\right) = 87.1^\circ - 21.8^\circ = 65.3^\circ$$

thus, the system response to the input $\cos 2t$ is

$$y(t) = 0.372 \cos\left(2t + 65.3^\circ\right)$$

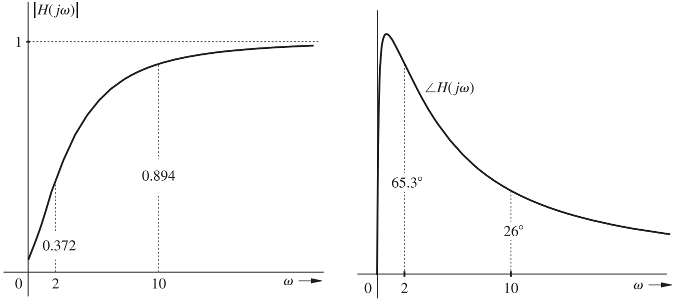


(b) for the input $\cos(10t - 50°)$, we have

$$|H(j10)| = 0.894 \quad \text{and} \quad \angle H(j10) = 26°$$

therefore, the system response $y(t)$ is

$$y(t) = 0.894\cos(10t - 50° + 26°) = 0.894\cos(10t - 24°)$$



amplitude response shows that the system has highpass filtering characteristics

Plotting frequency response using MATLAB

$$H(s) = \frac{s + 0.1}{s + 5}$$

Method I: use anonymous function to define the transfer function $H(s)$

```
>> H = @(s) (s+0.1)./(s+5); omega = 0:.01:20;
>> subplot(1,2,1); plot(omega,abs(H(1j*omega)),'k-');
>> subplot(1,2,2); plot(omega,angle(H(1j*omega))*180/pi,'k-');
```

Method II: use the `freqs` command to compute frequency response

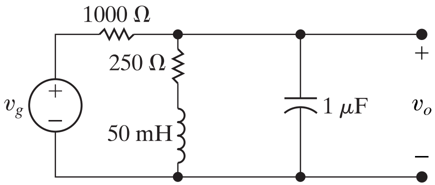
```
>> B = [1 0.1]; A = [1 5]; omega = 0:.01:20; H = freqs(B,A,omega);
>> subplot(1,2,1); plot(omega,abs(H),'k-');
>> subplot(1,2,2); plot(omega,angle(H)*180/pi,'k-');
```

both approaches generate plots that match the previous example

Example 8.9

find the steady-state expression for v_o given that the input voltage is sinusoidal

$$v_g = 120\cos(5000t + 30^\circ)\,\mathrm{V}$$



Solution: computing the transfer function using circuit analysis:

$$H(s) = \frac{V_o(s)}{V_g(s)} = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}$$

the frequency of the voltage source is $5000 \mathrm{rad/s}$ and

$$\begin{aligned} H(j5000) &= \frac{1000(5000 + j5000)}{-25 * 10^6 + j5000(6000) + 25 \times 10^6} \\ &= \frac{1 + j1}{j6} = \frac{1 - j1}{6} = \frac{\sqrt{2}}{6}\angle -45^\circ \end{aligned}$$

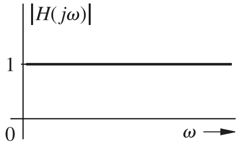
thus

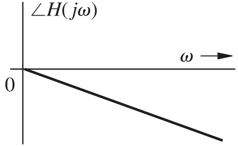
$$\begin{aligned} v_{o_{ss}} &= \frac{(120)\sqrt{2}}{6} \cos\left(5000t + 30^\circ - 45^\circ\right) \\ &= 20\sqrt{2} \cos\left(5000t - 15^\circ\right) \mathrm{V} \end{aligned}$$

Ideal delay frequency response

ideal delay of T seconds $H(s) = e^{-sT}$:

$$|H(j\omega)| = 1 \quad \text{and} \quad \angle H(j\omega) = -\omega T$$



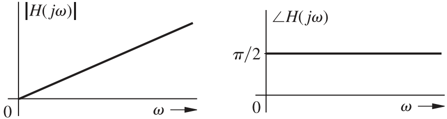


- if the input is $\cos \omega t$, the output is $\cos \omega(t - T)$
- the amplitude response (gain) is unity for all frequencies
- the phase response is linearly proportional to the frequency ω with a slope $-T$

Ideal differentiator frequency response

ideal differentiator $H(s) = s$:

$$|H(j\omega)| = \omega \quad \text{and} \quad \angle H(j\omega) = \frac{\pi}{2}$$

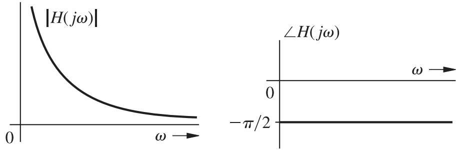


- for input $\cos \omega t$, the output is $\omega \cos[\omega t + (\pi/2)] = -\omega \sin \omega t$
- the amplitude response (gain) increases linearly with frequency ω
- the output sinusoid undergoes a phase shift $\pi/2$ with respect to the input $\cos \omega t$
- since $|H(j\omega)| = \omega$, higher-frequency components are magnified
- a differentiator can increase the noise is a signal, which is undesirable

Ideal integrator frequency response

an ideal integrator $H(s) = \frac{1}{s}$:

$$|H(j\omega)| = \frac{1}{\omega} \quad \text{and} \quad \angle H(j\omega) = -\frac{\pi}{2}$$



- if the input is $\cos \omega t$, the output is $(1/\omega)\sin \omega t = (1/\omega)\cos[\omega t - (\pi/2)]$
- amplitude response is proportional to $1/\omega$, and phase response is constant $-\pi/2$
- because $|H(j\omega)| = 1/\omega$, the ideal integrator suppresses higher-frequency components and enhances lower-frequency components with ω
- rapidly varying noise signals are suppressed (smoothed out) by an integrator

Steady-state response to causal sinusoidal inputs

for the input $x(t) = e^{j\omega t}u(t)$, we have (assume distinct roots)

$$Y(s) = X(s)H(s) = X(s)\frac{P(s)}{Q(s)} = \frac{P(s)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_N)(s-j\omega)}$$

$$= \sum_{i=1}^{n} \frac{k_i}{s-\lambda_i} + \frac{H(j\omega)}{s-j\omega}$$

for some constants k_i; taking inverse Laplace transform:

$$y(t) = \underbrace{\sum_{i=1}^{n} k_i e^{\lambda_i t}u(t)}_{\text{transient component } y_{\text{tr}}(t)} + \underbrace{H(j\omega)e^{j\omega t}u(t)}_{\text{steady-state component } y_{\text{ss}}(t)}$$

- for stable system, the characteristic mode terms $e^{\lambda_i t}$ goes to zero
- for a causal sinusoidal input $\cos(\omega t)u(t)$, the steady-state response is

$$y_{\text{ss}}(t) = |H(j\omega)| \cos[\omega t + \angle H(j\omega)]u(t)$$

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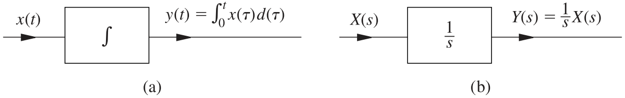
System realization

system realization is the process of putting together system components to form an overall system with a desired transfer function

Transfer function realization ($M = N$th-order)

$$H(s) = \frac{b_0 s^N + b_1 s^{N-1} + \cdots + b_{N-1} s + b_N}{s^N + a_1 s^{N-1} + \cdots + a_{N-1} s + a_N} \tag{8.1}$$

- can be realized by using integrators or differentiators with adders and multipliers
- in frequency domain realization, the integrator can be represented as $1/s$:



- integrators can be modeled using op-amp circuits

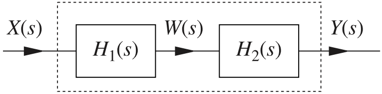
Example: consider the specific case:

$$H(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} = \frac{b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}}{1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}}$$

to use integrators, we express $H(s)$ as

$$H(s) = \underbrace{\left(b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}\right)}_{H_1(s)} \underbrace{\left(\frac{1}{1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}}\right)}_{H_2(s)}$$

we can realize $H(s)$ as a cascade of $H_1(s)$ followed by $H_2(s)$



- we have

$$W(s) = H_1(s)X(s) = \left(b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}\right) X(s)$$

signal $W(s)$ can be obtained by successive integration of the input $x(t)$

- we have $Y(s) = H_2(s)W(s)$, hence:

$$W(s) = \left(1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}\right) Y(s)$$

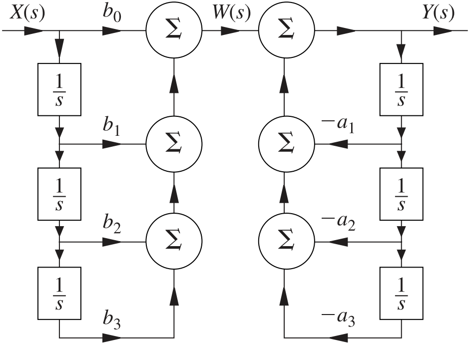
rearranging

$$Y(s) = W(s) - \left(\frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}\right) Y(s)$$

to obtain $Y(s)$, we subtract $a_1Y(s)/s, a_2Y(s)/s^2$, and $a_3Y(s)/s^3$ from $W(s)$

successive integration of $Y(s)$ yields $Y(s)/s, Y(s)/s^2$, and $Y(s)/s^3$

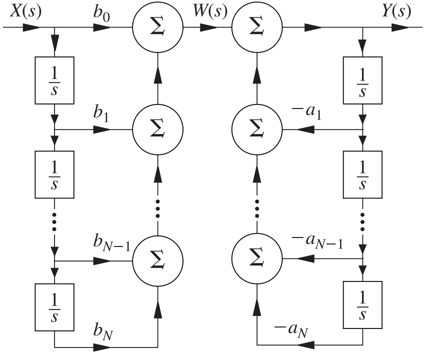
putting things together, $Y(s)$ can be synthesized (realized) as



left-half section represents $H_1(s)$ and the right-half is $H_2(s)$

Direct form I

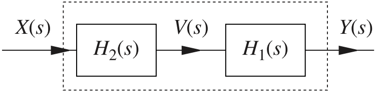
the *direct form I* (DFI) realization to equation (8.1) for any value of N



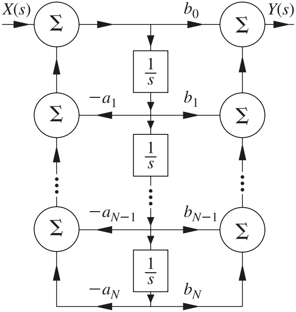
this realization requires $2N$ integrators to realize an Nth-order transfer function

Canonic direct form II

we can also realize $H(s)$ where $H_2(s)$ is followed by $H_1(s)$



doing so gives the canonic DFII or the **canonic direct form**:



a canonic realization has N integrators, which equals order of system

Example 8.10

find the canonic direct form realization of the following transfer functions:

(a) $\dfrac{5}{s+7}$

(b) $\dfrac{s}{s+7}$

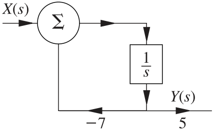
(c) $\dfrac{s+5}{s+7}$

(d) $\dfrac{4s+28}{s^2+6s+5}$

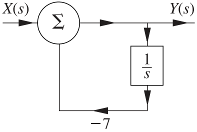
Solution:

(a) the transfer function is of the first order $(N = 1)$; therefore, we need only one integrator for its realization; the feedback and feedforward coefficients are

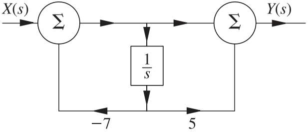
$$a_1 = 7 \quad \text{and} \quad b_0 = 0, \quad b_1 = 5$$



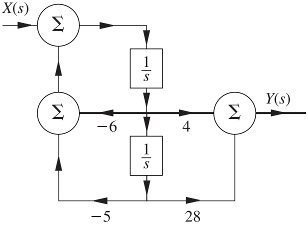
(b) we have $a_1 = 7$, $b_0 = 1$, and $b_1 = 0$



(c) here $H(s)$ is a first-order transfer function with $a_1 = 7$ and $b_0 = 1, b_1 = 5$



(d) this is a second-order system with $b_0 = 0, b_1 = 4, b_2 = 28, a_1 = 6$, and $a_2 = 5$;



Cascade and parallel realizations

an Nth-order transfer function $H(s)$ can be realized as a cascade (series) or parallel form of these N first-order transfer functions

Example:

$$H(s) = \frac{4s + 28}{s^2 + 6s + 5}$$

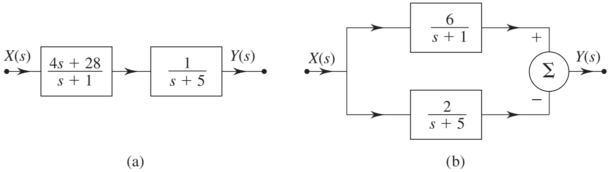
we can express $H(s)$ as

$$H(s) = \frac{4s + 28}{(s+1)(s+5)} = \underbrace{\left(\frac{4s+28}{s+1}\right)}_{H_1(s)} \underbrace{\left(\frac{1}{s+5}\right)}_{H_2(s)}$$

we can also express $H(s)$ as a sum of partial fractions as

$$H(s) = \frac{4s + 28}{(s+1)(s+5)} = \underbrace{\frac{6}{s+1}}_{H_3(s)} - \underbrace{\frac{2}{s+5}}_{H_4(s)}$$

these equations give us the option of realizing $H(s)$ as a cascade of $H_1(s)$ and $H_2(s)$ or a parallel of $H_3(s)$ and $H_4(s)$



- each of the first-order transfer functions can be implemented by using canonic direct realizations, discussed earlier
- many different ways to realize a system (*e.g.*, different ways of grouping the factors)

Realizations of complex conjugate poles

the complex poles in $H(s)$ should be realized as a second-order (quadratic) factor because we cannot implement multiplication by complex numbers

Example:

$$H(s) = \frac{10s + 50}{(s+3)\,(s^2 + 4s + 13)} = \frac{2}{s+3} - \frac{1+j2}{s+2-j3} - \frac{1-j2}{s+2+j3}$$

to realize the above, we can create a cascade realization from $H(s)$:

$$H(s) = \left(\frac{10}{s+3}\right)\left(\frac{s+5}{s^2+4s+13}\right)$$

or, we can create a parallel realization from $H(s)$ expressed in sum form as

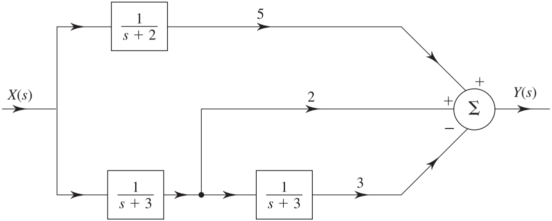
$$H(s) = \frac{2}{s+3} - \frac{2s-8}{s^2+4s+13}$$

Example 8.11

determine the parallel realization with least amount of integrators of

$$H(s) = \frac{7s^2 + 37s + 51}{(s+2)(s+3)^2} = \frac{5}{s+2} + \frac{2}{s+3} - \frac{3}{(s+3)^2}$$

Solution: observe that the terms $1/(s+3)$ and $1/(s+3)^2$ can be realized with a cascade of two subsystems, each having a transfer function $1/(s+3)$



each of the three transfer functions can realized as in the previous example

Transposed realization

transposed realization is equivalent t a given realization, generated as follows

1. reverse all the arrow directions without changing the scalar multiplier values
2. replace pickoff nodes by adders and vice versa
3. replace the input $X(s)$ with the output $Y(s)$ and vice versa

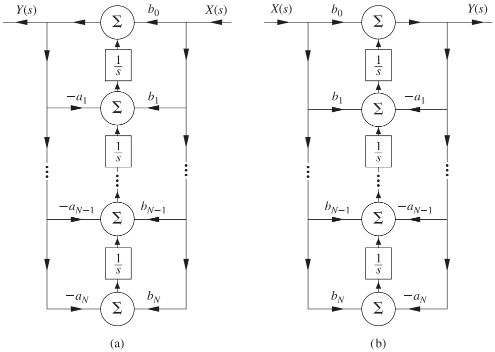


fig. (b) is fig. (a) reoriented in the conventional form

Example 8.12

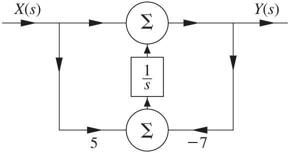
find the transpose canonic direct realizations of

(a) $\dfrac{s+5}{s+7}$

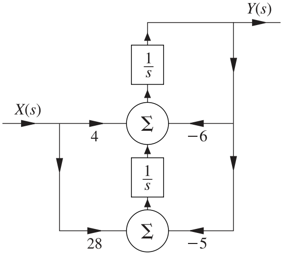
(b) $\dfrac{4s+28}{s^2+6s+5}$

Solution:

(a) here, $N = 1$ with $a_1 = 7, b_0 = 1, b_1 = 5$



(b) in this case, $N = 2$ with $b_0 = 0, b_1 = 4, b_2 = 28, a_1 = 6, a_2 = 5$

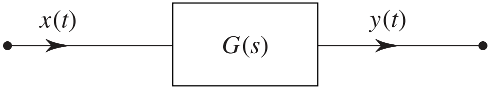


Outline

- the transfer function
- stability
- frequency response
- LTI systems realization
- **introduction to feedback system design***

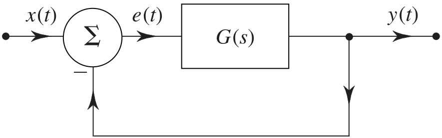
System design

systems aim to produce a specific output $y(t)$ for an input $x(t)$



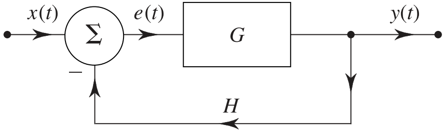
- open-loop systems should yield the desired output but may change due to aging, component replacement, or environment
- these variations can alter the output, requiring corrections at the input
- the needed input correction is the difference between actual and desired output
- feedback of the output or its function to the input may counteract variations

Feedback



- address problems from disturbances like noise signals or environmental changes
- aim to meet objectives within tolerances adapting to system changes
- allows supervision and self-correction against parameter variations/disturbances

Example: negative feedback amplifier



- forward amplifier gain $G = 10,000$, with $H = 0.01$ feedback, gives:

$$T = \frac{G}{1 + GH} = \frac{10,000}{1 + 100} = 99.01$$

- if G changes to $20,000$, the new gain is:

$$T = \frac{20,000}{1 + 200} = 99.5$$

- shows reduced sensitivity to forward gain G variations
- changing G by 100% changes T by 0.5%

Example: positive feedback amplifier

$$T = \frac{G}{1 - GH}$$

- for $G = 10,000$ and $H = 0.9 \times 10^{-4}$, gain T is:

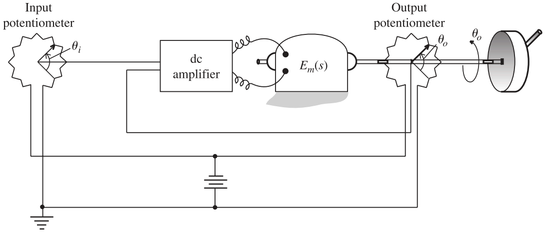
$$T = \frac{10,000}{1 - 0.9\,(10^4)\,(10^{-4})} = 100,000$$

- with $G = 11,000$, new gain is:

$$T = \frac{11,000}{1 - 0.9(11,000)\,(10^{-4})} = 1,100,000$$

- highlights sensitivity to forward gain G changes
- positive feedback increases system gain but also sensitivity to parameter changes, leading to potential instability
- for $G = 111,111$, $GH = 1$ results in $T = \infty$ and system instability

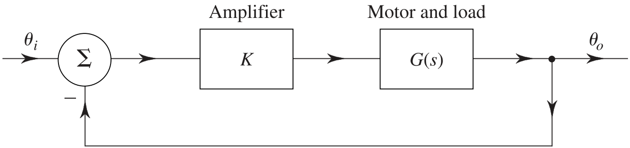
Automatic position system



controls the angular position of heavy objects like tracking antennas or gun mounts

- input θ_i is the desired angular position
- actual position θ_o measured by a potentiometer
- difference $\theta_i - \theta_o$ amplified and applied to motor input
- motor stops if $\theta_i - \theta_o = 0$, moves if $\theta_o \neq \theta_i$
- system controls remote object's angular position by setting input potentiometer

Block diagram of automatic position system



- amplifier gain is K (adjustable)
- motor transfer function $G(s)$ relates output angle θ_o to input voltage
- system transfer function $T(s) = \frac{KG(s)}{1+KG(s)}$
- next, we examine behavior for step and ramp inputs

Step response

- step input indicates instantaneous angle change
- we want to assess transient time to reach desired angle
- output $\theta_o(t)$ found for input $\theta_i(t) = u(t)$
- step input test reveals system's performance under various conditions

for step input $\theta_i(t) = u(t), \Theta_i(s) = \frac{1}{s}$,

$$\Theta_o(s) = \frac{KG(s)}{s[1 + KG(s)]}$$

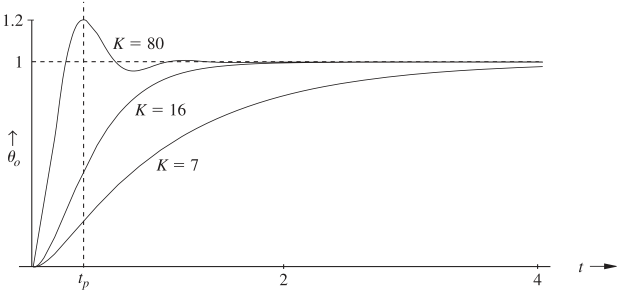
assuming $G(s) = \frac{1}{s(s+8)}$, investigate system behavior for different K values

$$\Theta_o(s) = \frac{\frac{K}{s(s+8)}}{s\left[1 + \frac{K}{s(s+8)}\right]} = \frac{K}{s\left(s^2 + 8s + K\right)}$$

we have

$$\theta_o(t) = \left(1 - \tfrac{7}{6}e^{-t} + \tfrac{1}{6}e^{-7t}\right) u(t), \quad K = 7$$
$$\theta_o(t) = \left[1 + \tfrac{\sqrt{5}}{2}e^{-4t} \cos(8t + 153^\circ)\right] u(t), \quad K = 80$$

response for $K = 80$ reaches final position faster but with high overshoot/oscillations



for $K = 80$

- percent overshoot (PO) is 21%; peak time $t_p = 0.393$, rise time $t_r = 0.175$
- steady-state error is zero, settling time $t_s \approx 1$ second
- a good system has small overshoot, t_r, t_s, and steady-state error

to avoid oscillations in an automatic position system, choose real characteristic roots

- characteristic polynomial is $s^2 + 8s + K$
- for $K > 16$, roots are complex; for $K < 16$, roots are real
- fastest response without oscillations at $K = 16$
- system is
  - critically damped at $K = 16$
  - underdamped if $K > 16$
  - overdamped if $K < 16$

for $K = 16$,

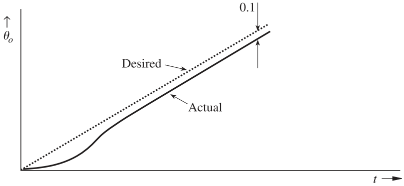
$$\begin{aligned}\Theta_o(s) &= \frac{16}{s(s^2 + 8s + 16)} = \frac{16}{s(s+4)^2} \\ &= \frac{1}{s} - \frac{1}{s+4} - \frac{4}{(s+4)^2} \\ \theta_o(t) &= \left[1 - (4t + 1)e^{-4t}\right] u(t)\end{aligned}$$

Ramp response

response of to a ramp input $\theta_i(t) = tu(t)$ or $\Theta_i(s) = \frac{1}{s^2}$ when $K = 80$

$$\Theta_o(s) = \frac{80}{s^2(s^2+8s+80)} = -\frac{0.1}{s} + \frac{1}{s^2} + \frac{0.1(s-2)}{s^2+8s+80}$$

$$\theta_o(t) = \left[-0.1 + t + \tfrac{1}{8}e^{-8t}\cos(8t + 36.87°)\right]u(t)$$



response to a ramp input $\theta_i(t) = tu(t)$ with a steady-state error

- steady-state error $e_r = 0.1$ radian may be tolerable
- zero error requires compensator addition

Matlab example

using feedback system $G(s) = \frac{K}{s(s+8)}$ and $H(s) = 1$, determine step response for $K = 7, 16, 80$

- code for unit step response

```
H = tf(1,1); K = 7; G = tf([K],conv([1 0],[1 8])); Ha = feedback(G,H);
H = tf(1,1); K = 16; G = tf([K],conv([1 0],[1 8])); Hb = feedback(G,H);
H = tf(1,1); K = 80; G = tf([K],conv([1 0],[1 8])); Hc = feedback(G,H);
clf; step(Ha,'k-',Hb,'k--',Hc,'k-.');
legend('K = 7','K = 16','K = 80','Location','best');
```

- code for unit ramp response when $K = 80$

```
t = 0:.001:1.5; Hd = series(Hc,tf([1],[1 0]));
step(Hd,'k-',t); title('Unit Ramp Response');
```

Design specification

- transient specifications: overshoot, rise time, settling time for step input
- steady-state error: difference between desired and actual response in steady state
- sensitivity to system parameter variations or disturbances
- system stability under operating conditions

References

- B. P. Lathi, *Linear Systems and Signals*, Oxford University Press.