4. Discrete-time signals

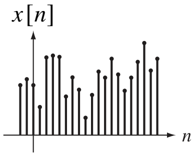
- DT signals
- signal operations
- useful DT signals
- signal energy and power

Discrete-time signals

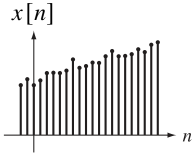
a *discrete-time (DT) signal* is a function defined over an *integer* variable

$$x[n] \quad \text{where} \quad n \in \{\ldots, -1, 0, 1, \ldots\}$$

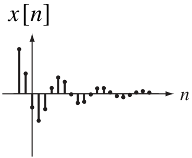
- a sequence of numbers $\ldots, x[-1], x[0], x[1], \ldots$
- a CT signal $x(t)$ can discretized by sampling it $x[n] = x(t_n)$ over discrete instants $\{t_n\}$, $n = 0, 1, 2, \ldots$
- examples:



stock market daily averages



weekly average tempratures



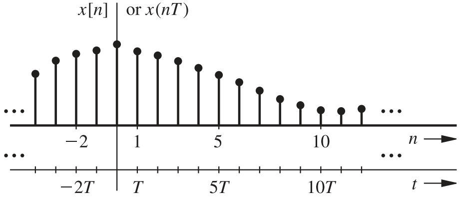
samples from exponentially damped sinusoid

Uniform sampling

uniform sampling a continuous-time signal $x(t)$ gives a DT signal:

$$x[n] = x(nT)$$

- n is an integer
- T is *sampling period* or *sampling interval*



Example: sampling $x(t) = e^{-t}$ with $T = 0.1$:

$$x[n] = e^{-nT} = e^{-0.1n} \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots$$

Causal and periodic signals

Causal signals: $x[n]$ is *causal* if $x[n] = 0$ for $n < 0$

- a signal $x[n]$ is *anticausal* if $x[n] = 0, \ n \geq 0$
- a signal that has values before $n = 0$ is called *noncausal*

Periodic signals: a signal $x[n]$ is *periodic* if for some positive constant N:

$$x[n] = x[n + N], \quad \text{for all } n$$

- *fundamental period* N_0 is the smallest N such that the above holds
- fund. frequency is $F_0 = 1/N_0$ cycles/sample and $\Omega_0 = 2\pi/N_0$ rad/sample
- a periodic signal must start at $n = -\infty$ and continue forever

Discrete-time sinusoid

$$A\cos(\Omega n + \theta) = A\cos(2\pi F n + \theta)$$

- A is the *amplitude*, θ is the *phase* in radians
- *frequency* Ω has dimension *radians per sample*
- $F = \Omega/2\pi$ with dimension *cycles (radians/2π) per sample*
- uniform sampling of $x(t) = \cos\omega t$ with sampling rate T seconds gives

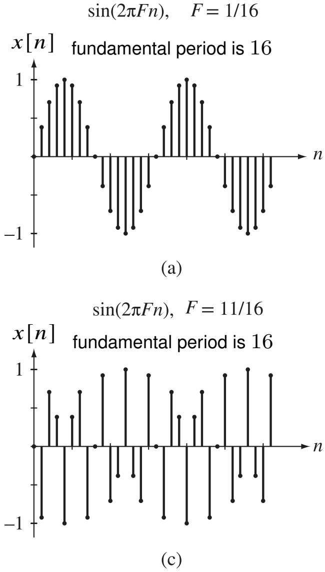
$$x[n] = \cos(\omega n T) = \cos(\Omega n) \quad \text{where} \quad \Omega = \omega T$$

- DT sinsuoids are not necessarily periodic

Periodicity of DT sinusoid: $\cos(\Omega n) = \cos(2\pi F n)$ is periodic if $\Omega N = 2\pi m$ for some non-zero integers m and N

- implies DT sinusoid is periodic if $F = m/N$ is a rational number
- if $F = m_0/N_0$ expressed in simplest form, then N_0 is the *fundamental period*

Examples



sin(2πFn), F = 1/16
x[n]
fundamental period is 16
1
−1
n
(a)
sin(2πFn), F = 2/16
x[n]
fundamental period is 8
1
−1
n
(b)
sin(2πFn), F = 11/16
x[n]
fundamental period is 16
1
−1
n
(c)
sin(2πFn), F = π/16
x[n]
aperiodic
1
−1
n
(d)

Sum periodic signals

the sum of periodic DT signals is always periodic

- let $x_1[n]$ and $x_2[n]$ be periodic with fundamental periods N_{01} and N_{02} and

$$x[n] = x_1[n] + x_2[n]$$

- $x[n]$ is periodic and the fundamental period is:

$$N_0 = \text{LCM}(N_{01}, N_{02}) = qN_{01} = pN_{02}$$

where $N_{01}/N_{02} = p/q$ for some integers p and q in smallest form

Example:

$$x[n] = 2\cos(9\pi n/4) - 3\sin(6\pi n/5)$$

we can write the function as

$$x[n] = 2\cos(2\pi(9/8)n) - 3\sin(2\pi(3/5)n)$$

we have $N_{01} = 8$ and $N_{02} = 5$; hence $x[n]$ is periodic and $N_0 = \text{LCM}(8, 5) = 40$

Discrete-time exponential

the *discrete-time exponential function* is

$$x[n] = \gamma^n$$

- can be expressed in usual form $\gamma^n = e^{\lambda n}$ where $\gamma = e^{\lambda}$
- for discrete-time signals, γ^n is preferred over $e^{\lambda n}$

Complex exponential: for complex $\gamma = re^{j\Omega}$, we get

$$x[n] = r^n e^{j\Omega n} = r^n(\cos\Omega n + j\sin\Omega n)$$

- the frequency is $|\Omega|$
- the angle is $n\Omega$
- in complex plane, $e^{j\Omega n}$ is a point on a unit circle at an angle Ωn

Nature of γ^n

for $\lambda = a + jb$, we have $e^{\lambda n} = \gamma^n$ where

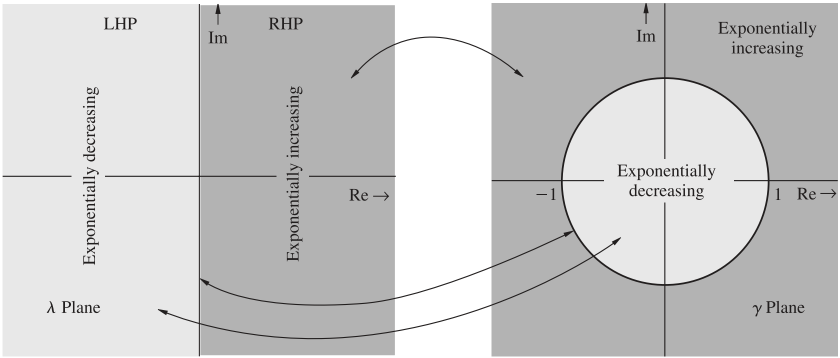
$$\gamma = e^{\lambda} = e^{a} e^{jb} \quad \text{and} \quad |\gamma| = |e^{a}||e^{jb}| = e^{a}$$

Nature of $e^{\lambda n}$

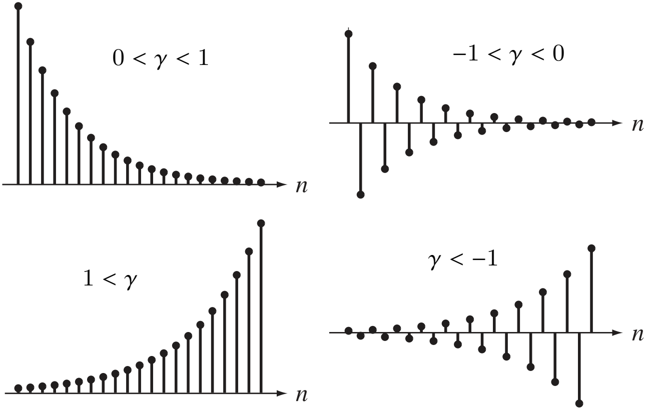
- $e^{\lambda n}$ grows exponentially with n if $\mathrm{Re}\,\lambda > 0$ (λ in RHP)
- $e^{\lambda n}$ decays exponentially with n if $\mathrm{Re}\,\lambda < 0$ (λ in LHP)
- $e^{\lambda n}$ constant or oscillate if $\mathrm{Re}\,\lambda = 0$ (λ on imaginary axis)

Nature of γ^n

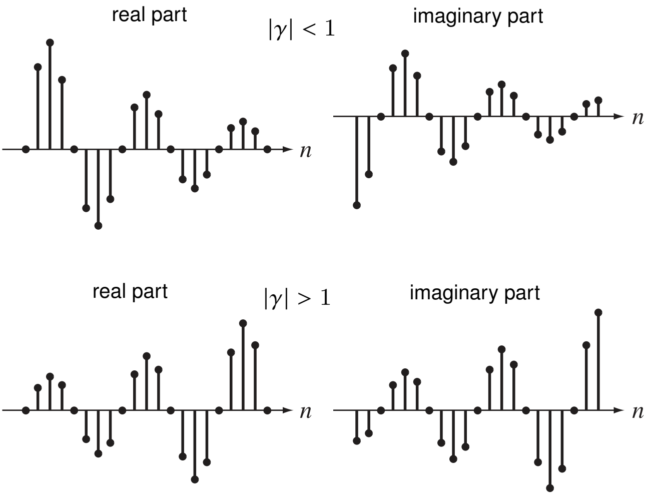
- γ^n grows exponentially with n if $|\gamma| > 1$ (γ outside unit circle)
- γ^n decays exponentially with n if $|\gamma| < 1$ (γ inside unit circle)
- γ^n is a constant or oscillate if $|\gamma| = 1$ (γ on unit circle)


LHP
Im
RHP
Exponentially decreasing
Exponentially increasing
Re →
λ Plane
Im
Exponentially
increasing
Exponentially
decreasing
−1
1
Re →
γ Plane

Behavior of γ^n for real γ



Behavior of γ^n for complex γ



Example: plotting DT signals in Matlab

use Matlab to plot the following signals

(a) $x_a[n] = (0.8)^n$ over $0 \leq n \leq 8$

(b) $x_b[n] = (-0.8)^n$ over $0 \leq n \leq 8$

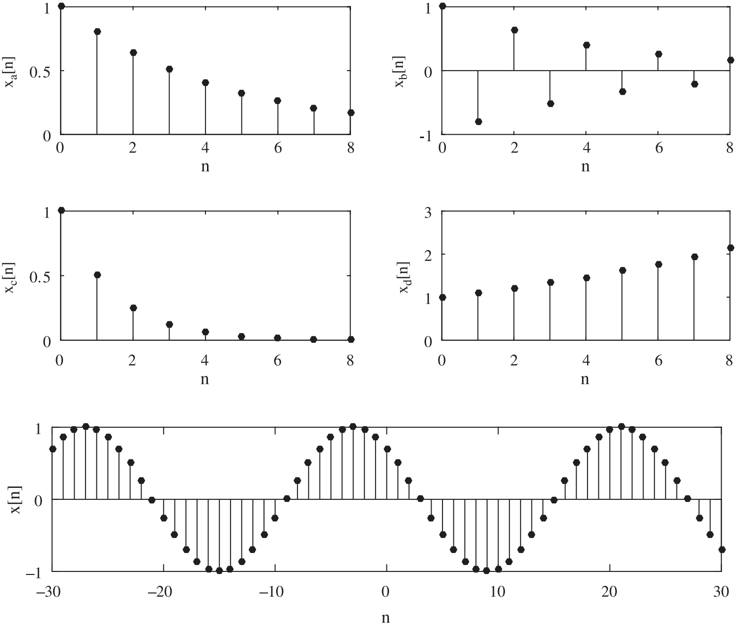
(c) $x_c[n] = (0.5)^n$ over $0 \leq n \leq 8$

(d) $x_d[n] = (1.1)^n$ over $0 \leq n \leq 8$

(e) $x[n] = \cos(\frac{\pi}{12}n + \frac{\pi}{4})$ over $-30 \leq n \leq 30$

`stem` command is used to plot DT signals

```
n = (0:8); x_a = @(n) (0.8).^n; x_b = @(n) (-0.8).^(n);
x_c = @(n) (0.5).^n; x_d = @(n) (1.1).^n;
subplot(2,2,1); stem(n,x_a(n),'filled','k'); ylabel('x_a[n]'); xlabel('n');
subplot(2,2,2); stem(n,x_b(n),'filled','k'); ylabel('x_b[n]'); xlabel('n');
subplot(2,2,3); stem(n,x_c(n),'filled','k'); ylabel('x_c[n]'); xlabel('n');
subplot(2,2,4); stem(n,x_d(n),'filled','k'); ylabel('x_d[n]'); xlabel('n');
figure
n = (-30:30); x = @(n) cos(n*pi/12+pi/4);
clf; stem(n,x(n),'filled','k'); ylabel('x[n]'); xlabel('n');
```


x_a[n]
x_b[n]
x_c[n]
x_d[n]
x[n]
n
0
0.5
1
-1
2
3
4
6
8
–30
–20
–10
10
20
30

Outline

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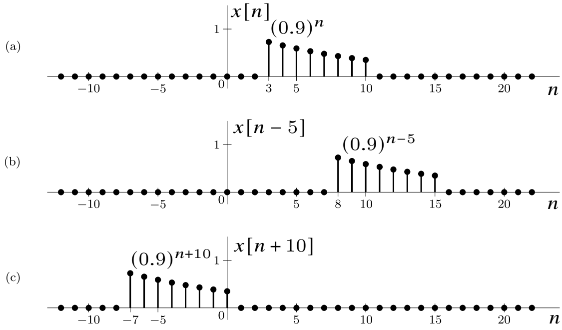
- **signal operations**
- useful DT signals
- signal energy and power

Time-shifting

the signal $x[n]$ can be *time shifted* to the right or left by $n_0 > 0$ units

$$x[n - n_0] \qquad \text{(right-shifted (delayed) signal)}$$
$$x[n + n_0] \qquad \text{(left-shifted (advanced) signal)}$$

Example:

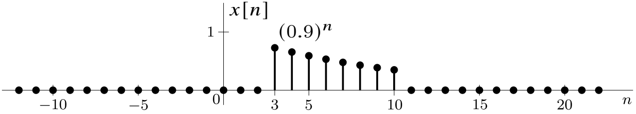


Time reversal

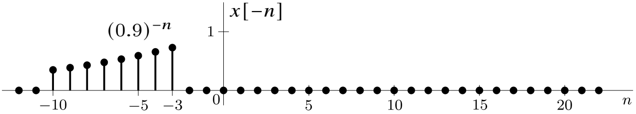
the *time reversal* operation $x[-n]$ rotates $x[n]$ about the vertical axis

Example:

(a)



(b)



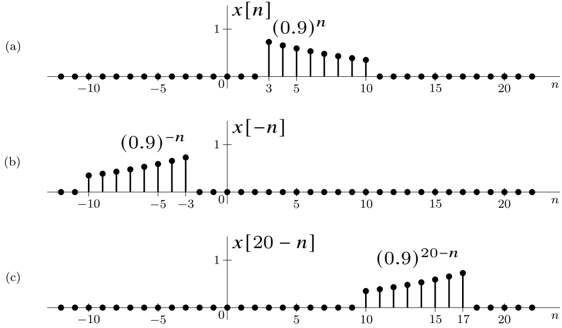
Time-reversal and shifting

the time-reversal and shifting operation is $x[k-n]$

1. $x[n] \xrightarrow{\text{time reverse}} x[-n] \xrightarrow{\text{(right) shift by } k} x[-(n-k)] = x[k-n]$

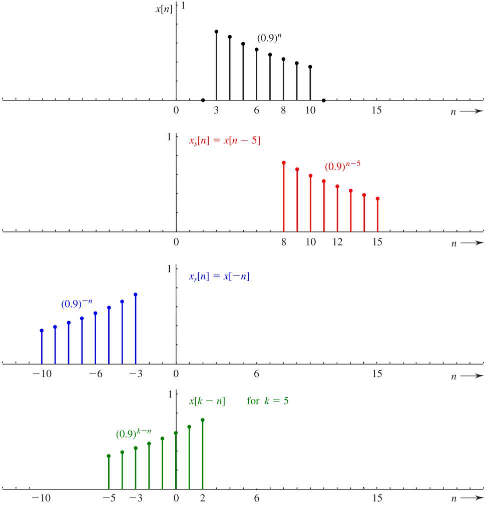
2. $x[n] \xrightarrow{\text{(left) shift by } k} x[n+k] \xrightarrow{\text{time reverse}} x[k-n]$

Example: find $x[20-n]$



Example 4.1

plot $x[5-n]$ for the signal $x[n]$



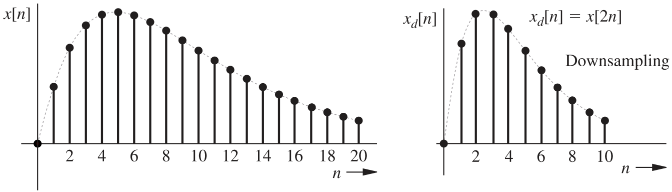
Time scale: downsampling (time compression)

downsampling is the compression of $x[n]$ by integer factor M:

$$x_d[n] = x[Mn]$$

- $x[Mn]$ selects every Mth sample: $x[0], x[M], x[2M], \ldots$
- reduces the number of samples by factor M (loss of samples)
- if $x[n]$ is sampled CT signal, this operation reduces sampling rate by M

Example:



Time scale: upsampling

upsampling is the expansion of $x[n]$ by integer factor L

$$x_e[n] = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{otherwise } (n/L \text{ noninteger}) \end{cases}$$

- for $n = 0, 1, 2, \ldots$, $x_e[n]$ is:

$$x[0], \underbrace{0, 0, \ldots, 0}_{L-1 \text{ zeros}}, x[1], \underbrace{0, 0, \ldots, 0}_{L-1 \text{ zeros}}, x[2], \underbrace{0, 0, \ldots, 0}_{L-1 \text{ zeros}}, x[3], \ldots$$

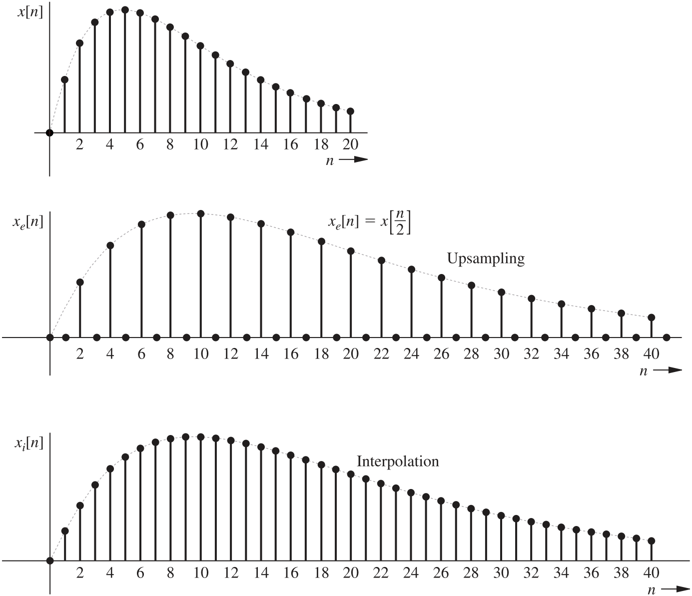
- the sampling rate of $x_e[n]$ is L times that of $x[n]$

Interpolation

- the process of filling-in the zero-valued samples is called *interpolation*
- example: *linear interpolation* for $L = 2$, we replace the zero samples by:

$$x_i[n] = \frac{1}{2}(x_e[n-1] + x_e[n+1])$$

Example



$x[n]$
2 4 6 8 10 12 14 16 18 20
n
$x_e[n]$
$x_e[n] = x\left[\frac{n}{2}\right]$
Upsampling
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
n
$x_i[n]$
Interpolation
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
n

Combined operations

$$x[an-b] \quad \text{where } a \text{ and } b \text{ are integers}$$

Method 1:

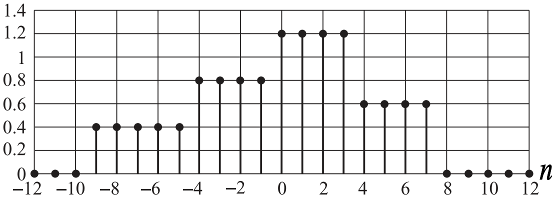
$$x[n] \overset{\text{time shift by } b}{\longrightarrow} x[n-b] \overset{\text{time scale by } a}{\longrightarrow} x[an-b]$$

Method 2: if b/a is an integer, then $x[an-b] = x[a(n-b/a)]$

$$x[n] \overset{\text{time scale by } a}{\longrightarrow} x[an] \overset{\text{time shift by } b/a}{\longrightarrow} = x[a(n-b/a)] = x[an-b]$$

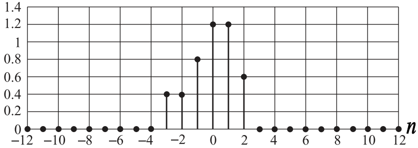
Example 4.2

sketch $x[-15 - 3n]$ for the DT signal shown below

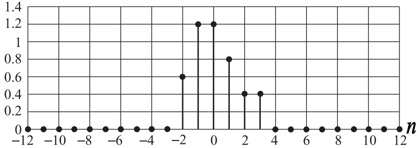


Solution: we write $x[-15 - 3n] = x[-3(n+5)]$ and follow the steps given next

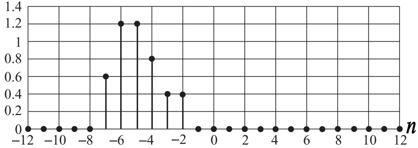
- compress $x[n]$ by 3 to get $x[3n]$



- time-reverse $x[3n]$ to get $x[-3n]$

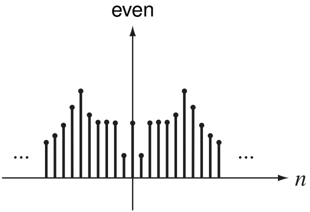


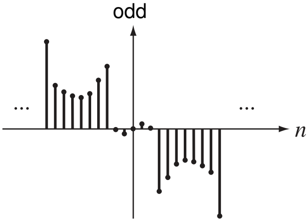
- left-shift $x[-3n]$ by 5 to obtain $x[-3(n+5)] = x[-15-3n]$



Even and odd signals

- $x_e[n]$ is **even** if $x_e[n] = x_e[-n]$
- $x_o[n]$ is **odd** if $x_o[n] = -x_o[-n]$





every signal $x[n]$ can expressed as

$$x[n] = \underbrace{\frac{1}{2}\left[x[n] + x[-n]\right]}_{\text{even}} + \underbrace{\frac{1}{2}\left[x[n] - x[-n]\right]}_{\text{odd}}$$

Example 4.3

find the even and odd parts of the function, $x[n] = \sin(2\pi n/7)\left(1+n^2\right)$

Solution: the even part is

$$x_e[n] = \frac{\sin(2\pi n/7)\left(1+n^2\right) + \sin(-2\pi n/7)\left(1+(-n)^2\right)}{2} = 0$$

the odd part is

$$\begin{aligned} x_o[n] &= \frac{\sin(2\pi n/7)\left(1+n^2\right) - \sin(-2\pi n/7)\left(1+(-n)^2\right)}{2} \\ &= \sin(2\pi n/7)(1+n^2) \end{aligned}$$

the function is odd since the even part is zero

Properties

Multiplications

$$\text{even function} \times \text{odd function} = \text{odd function}$$

$$\text{odd function} \times \text{odd function} = \text{even function}$$

$$\text{even function} \times \text{even function} = \text{even function}$$

Symmetric summation of even function: for positive integer N

$$\sum_{n=-N}^{N} x[n] = x[0] + 2\sum_{n=1}^{N} x[n] \qquad (x[n] \text{ is even})$$

Symmetric summation of odd function: for positive integer N

$$\sum_{n=-N}^{N} x[n] = 0 \qquad (x[n] \text{ is odd})$$

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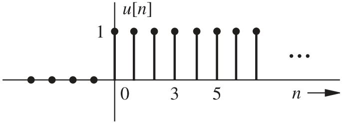
Unit step and unit ramp

(discrete-time) unit-step

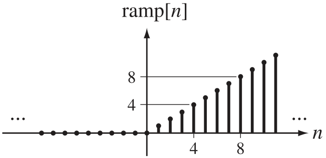
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0, & n < 0 \end{cases}$$



also called *unit-step sequence*

(discrete-time) unit-ramp

$$\mathrm{ramp}[n] = \begin{cases} n & n > 0 \\ 0, & n \leq 0 \end{cases} = nu[n]$$

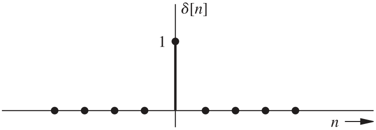


also called *unit-ramp sequence*

Unit impulse

(discrete-time) unit-impulse

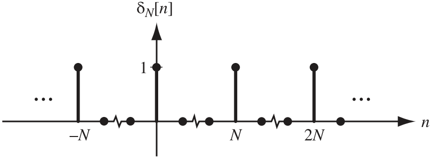
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



- also called *unit sample function* or *Kronecker delta function*
- defined everywhere (unlike continuous case)
- $\delta[n] = \delta[an]$ for any integer $a \neq 0$

unit periodic impulse (impulse train)

$$\delta_N[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$$



Properties

- *multiplication by DT impulse:*

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

- *sampling or sifting property:*

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Relation between unit step and unit impulse

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$\delta[n] = u[n] - u[n-1]$$

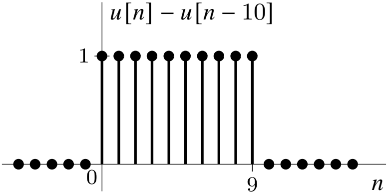
Rectangular sequence

the function

$$u[n-n_1]-u[n-n_2]$$

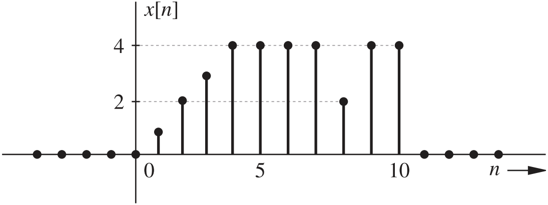
with $n_1 < n_2$ is a rectangular sequence from n_1 until (n_2-1)

Example:



Example 4.4

describe the signal $x[n]$ by a single expression valid for all n using unit-sequence



Solution: there are many ways to do this; one expression is

$$x[n] = n(u[n] - u[n-5]) + 4(u[n-5] - u[n-11]) - 2\delta[n-8]$$

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- **signal energy and power**

Energy signals

Energy of signal

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- if E_x is finite, the signal is called an *energy signal*
- E_x is finite if $|x[n]| \to 0$ as $|n| \to \infty$; infinite otherwise

Example: the energy of the signal $x[n] = (1/2)^n u[n]$ is

$$E_x = \sum_{n=-\infty}^{\infty} \left|\left(\tfrac{1}{2}\right)^n u[n]\right|^2 = \sum_{n=0}^{\infty} \left|\left(\tfrac{1}{2}\right)^n\right|^2 = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{2n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{4}\right)^n$$

using the formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, $|r| < 1$, we obtain

$$E_x = \frac{1}{1 - 1/4} = \frac{4}{3}$$

Power signals

Power of a signal

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- P_x is the time average (mean) of $|x[n]|^2$, also called *average power*
- $\sqrt{P_x}$ is the *rms* (root-mean-square) value of $x(t)$
- if P_x is finite and nonzero, the signal is called a *power signal*

Periodic signals power: a periodic signal $x[n]$ with period N_0 has power

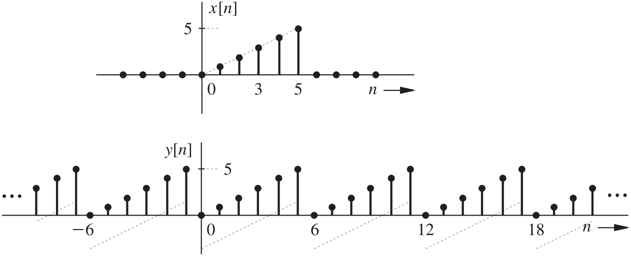
$$P_x = \frac{1}{N_0} \sum_{N_0} |x[n]|^2 = \frac{1}{N_0} \sum_{n=m_0}^{m_0+N_0-1} |x[n]|^2 \qquad \text{for any integer } m_0$$

Energy and power signals

- an energy signal has zero power
- a power signal has infinite energy
- hence, a signal cannot be both an energy signal and a power signal
- some signals are neither energy nor power signals

Example 4.5

find the energy of $x[n]$ and the power of the periodic signal $y[n]$ shown below



Solution:

$$E_x = \sum_{n=0}^{5} n^2 = 55$$

the period of signal y is $N_0 = 6$, hence

$$P_y = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |y[n]|^2 = \frac{1}{6} \sum_{n=0}^{5} n^2 = \frac{55}{6}$$

Example 4.6

find the energy E_x and power P_x of the signal $x[n] = 3\cos(\pi n/4)$

Solution: notice that $x[n]$ is 8-periodic and, therefore, a power signal:

$$P_x = \frac{1}{8}\sum_{n=0}^{7}|x[n]|^2 = \frac{1}{8}\left[2(3)^2 + 4(3/\sqrt{(2)})^2\right]$$
$$= \frac{1}{8}[18+18] = \frac{9}{2} = 4.5$$

since $0 < P_x < \infty$, we know that $E_x = \infty$

we can calculate the power in Matlab using the script:

```
x = @(n) 3*cos(pi*n/4); n = 0:7;
Px = sum(abs(x(n)).^2)/length(x(n))
```

[output is `Px = 4.5000`]

References

- B. P. Lathi, *Linear Systems and Signals*, Oxford University Press.
- M. J. Roberts, *Signals and Systems: Analysis Using Transform Methods and MATLAB*, McGraw Hill.