3. Time-domain analysis of continuous-time systems

- zero-input response
- impulse response
- convolution and zero-state response
- system stability

Linear systems response

response of linear system = ZIR + ZSR

Zero-input response (ZIR)

- output $y_0(t)$ due to initial conditions alone
- input is zero

Zero-state response (ZSR)

- output due to the input $x(t)$ alone
- all initial conditions are zero

Linear time-invariant (LTI) differential system

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t)$$
$$= b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{dx(t)}{dt} + b_M x(t)$$

- a_i and b_i are constants; we assume $M \leq N$ unless otherwise stated
- for $M > N$, the system acts as an $(M - N)$th order differentiator, which should be avoided since differentiation may greatly magnify high-frequency noise

Operator notation: using notation D^k for d^k/dt^k, we write

$$Q(D)y(t) = P(D)x(t)$$

where

$$Q(D) = D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N$$
$$P(D) = b_0 D^M + b_1 D^{M-1} + \cdots + b_{M-1} D + b_M$$

Zero-input response

for system in page 3.3, the ZIR is the solution to:

$$Q(D)y_0(t) = 0 \quad \text{or} \quad \left(D^N + a_1 D^{N-1} + \cdots + a_{N-1}D + a_N\right) y_0(t) = 0$$

- a linear combination of $y_0(t)$ and its N successive derivatives is zero for all t
- possible if $y_0(t)$ and all its N successive derivatives share the same form; only an exponential function $ce^{\lambda t}$ has this property
- suppose $y_0(t) = ce^{\lambda t}$ for some $c \neq 0$ and λ, then using

$$D^k y_0(t) = \frac{d^k y_0(t)}{dt^k} = c\lambda^k e^{\lambda t}, \quad k = 1, \ldots, N$$

we get

$$c\left(\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N\right) e^{\lambda t} = cQ(\lambda) = 0$$

hence, ce^{λ} is a ZIR if $Q(\lambda) = 0$

Characteristic equation

the *characteristic equation* of the differential system is

$$Q(\lambda) = \lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N = 0$$

- $Q(\lambda)$ is the *characteristic polynomial*
- we can express $Q(\lambda)$ in factorized form

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0$$

- the characteristic equation has N solutions $\lambda_1, \lambda_2, \ldots, \lambda_N$ called *characteristic roots* or *characteristic values* (also *eigenvalues*) of the system
- all $c_1 e^{\lambda_1 t}$, $c_2 e^{\lambda_2 t}, \ldots, c_N e^{\lambda_N t}$ satisfy the zero-input differential equation
- form of ZIR depends on whether the roots are distinct, repeated, and/or complex

Zero-input response

Distinct roots

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t}$$

- $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_N t}$ are the *characteristic* or *natural modes*
- $c_1, \ldots, c_N$ determined by N *auxiliary conditions* or *initial conditions*

Repeated roots: when the root λ_1 is repeated r times $\lambda_1 = \cdots = \lambda_r$, then

$$y_0(t) = \left(c_1 + c_2 t + \cdots + c_r t^{r-1}\right) e^{\lambda_1 t} + c_{r+1} e^{\lambda_{r+1} t} + \cdots + c_N e^{\lambda_N t}$$

- the characteristic modes are

$$e^{\lambda_1 t}, te^{\lambda_1 t}, \ldots, t^{r-1} e^{\lambda_1 t}, e^{\lambda_{r+1} t}, \ldots,\ e^{\lambda_N t}$$

- can be generalized to multiple repeated roots

Example 3.1

find the zero-input response, $y_0(t)$, of the LTIC systems described by

(a) $(D^2 + 3D + 2)y(t) = Dx(t)$ with $y_0(0) = 0$ and $\dot{y}_0(0) = -5$

(b) $(D^2 + 6D + 9)y(t) = (3D + 5)x(t)$ with $y_0(0) = 3$ and $\dot{y}_0(0) = -7$

Solution:

(a) the characteristic equation is

$$Q(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

characteristic roots are $\lambda_1 = -1, \lambda_2 = -2$ (characteristic modes are e^{-t}, e^{-2t})

therefore, the zero-input response has the form

$$y_0(t) = c_1 e^{-t} + c_2 e^{-2t}$$

taking derivative

$$\dot{y}_0(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

to find c_1 and c_2, we use initial conditions $y_0(0) = 0$ and $\dot{y}_0(0) = -5$:

$$y_0(0) = c_1 + c_2 = 0$$
$$\dot{y}_0(0) = -c_1 - 2c_2 = -5$$

solving gives $c_1 = -5$ and $c_2 = 5$; hence

$$y_0(t) = -5e^{-t} + 5e^{-2t}$$

(b) the characteristic equation is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$ and the characteristic roots are $\lambda_1 = -3, \lambda_2 = -3$ (repeated roots) (characteristic modes are e^{-3t}, te^{-3t})

the zero-input response has the form:

$$y_0(t) = (c_1 + c_2 t)\, e^{-3t}$$

using the initial conditions $y_0(0) = 3$ and $\dot{y}_0(0) = -7$, we can show that $c_1 = 3$ and $c_2 = 2$; hence,

$$y_0(t) = (3 + 2t)e^{-3t} \quad t \geq 0$$

Complex roots forms

for a real system (real coefficients of differential system), complex roots are conjugate pairs:

$$Q(\lambda) = (\lambda - [\alpha + j\beta])(\lambda - [\alpha - j\beta]) = 0$$

Complex form: treat as distinct roots

$$y_0(t) = c_1 e^{(\alpha + j\beta)t} + c_2 e^{(\alpha - j\beta)t}$$

for a real system, c_1 and c_2 are conjugates

$$c_1 = \frac{c}{2} e^{j\theta} \quad \text{and} \quad c_2 = \frac{c}{2} e^{-j\theta}$$

Real-form: we can rewrite the response equivalently as

$$y_0(t) = c e^{\alpha t} \cos(\beta t + \theta)$$

Example 3.2

find the zero-input response, $y_0(t)$, of the LTIC system described by

$$\left(D^2 + 4D + 40\right) y(t) = (D + 2)x(t)$$

with $y_0(0) = 2$ and $\dot{y}_0(0) = 16.78$

Solution: the characteristic equation is $\lambda^2 + 4\lambda + 40 = 0$; we can find the roots of a polynomial using MATLAB command:

```
r = roots([1 4 40])
```

[output: `r = -2.00+6.00i -2.00-6.00i`] hence the characteristic roots are complex $\lambda_1 = -2 + j6$ and $\lambda_2 = -2 - j6$; since $\alpha = -2$ and $\beta = 6$, the real-form solution is

$$y_0(t) = ce^{-2t}\cos(6t + \theta)$$

taking derivative, we get

$$\dot{y}_0(t) = -2ce^{-2t}\cos(6t + \theta) - 6ce^{-2t}\sin(6t + \theta)$$

to find c and θ, we use the initial conditions $y_0(0) = 2$ and $\dot{y}_0(0) = 16.78$:

$$2 = c\cos\theta$$
$$16.78 = -2c\cos\theta - 6c\sin\theta$$

solution of these two equations in two unknowns $c\cos\theta$ and $c\sin\theta$ is

$$c\cos\theta = 2 \quad \text{and} \quad c\sin\theta = -3.463$$

squaring and then adding these two equations yield

$$c^2 = (2)^2 + (-3.464)^2 = 16 \Longrightarrow c = 4$$

dividing $c\sin\theta = -3.463$ by $c\cos\theta = 2$, we have

$$\tan\theta = \frac{-3.463}{2} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{-3.463}{2}\right) = -\frac{\pi}{3}$$

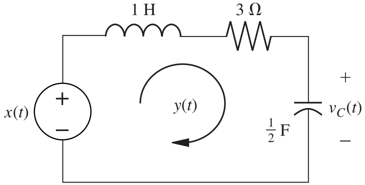
therefore,

$$y_0(t) = 4e^{-2t}\cos\left(6t - \frac{\pi}{3}\right)$$

Initial conditions

- in practice, the initial conditions are derived from the physical situation
- for example, in an RLC circuit, we may be given the conditions (initial capacitor voltages, initial inductor currents,...etc)

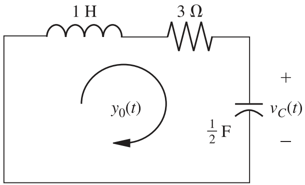
Example: find the ZIR $y_0(t)$ (loop current) for $t \geq 0$ if $y\left(0^-\right) = 0$ and $v_C\left(0^-\right) = 5$



the differential (loop) equation relating $y(t)$ to $x(t)$ is

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

to find ZIR $y_0(t)$, we set input to zero $x(t) = 0$



now the inductor current is still zero and the capacitor voltage is still 5 volts (cannot change instantaneously); thus, $y_0(0) = 0$; to determine $\dot{y}_0(0)$, note that

$$\dot{y}_0(0) + 3y_0(0) + v_C(0) = 0$$

since $y_0(0) = 0$ and $v_C(0) = 5$, we have $\dot{y}_0(0) = -5$

the problem reduces to finding zero-input response $y_0(t)$ of the system specified $(D^2 + 3D + 2)y(t) = 0$, with $y_0(0) = 0, \dot{y}_0(0) = -5$; from page 3.7, we have

$$y_0(t) = -5e^{-t} + 5e^{-2t} \quad t \geq 0$$

Meaning of 0^+ and 0^-

- conditions right before and after $t = 0$ are conditions at $t = 0^-$ and $t = 0^+$
- ZIR $y_0(t)$ does not depend on $x(t)$, hence $y_0(0^-) = y_0(0^+)$, $\dot{y}_0(0^-) = \dot{y}_0(0^+)$, $\ldots$
- in general, for the total response $y(t)$

$$y(0^-) \neq y(0^+), \dot{y}(0^-) \neq \dot{y}(0^+), \ldots$$

because of zero-state component (*i.e.*, input affects total response at 0^+)

Example: in the previous example, we have

$$\dot{y}(0^-) + 3y(0^-) + v_C(0^-) = 0 \quad \text{(at } t = 0^- \; x(t) = 0\text{)}$$
$$\dot{y}(0^+) + 3y(0^+) + v_C(0^+) = 10 \quad \text{(at } t = 0^+ \; x(t) = 10\text{)}$$

- current and capacitor voltage are $y(0^+) = y(0^-) = 0$, $v_C(0^+) = v_C(0^-) = 5$
- substituting these values into the above we have

$$y(0^-) = 0, \; y(0^+) = 0 \quad \text{and} \quad \dot{y}(0^-) = -5, \; \dot{y}(0^+) = 5$$

we see that $\dot{y}(0^-) \neq \dot{y}(0^+) = 5$ for the total response

Solving differential equations using MATLAB

$$(D^2 + 4D + k)y(t) = (3D + 5)x(t)$$

initial conditions $y_0(0) = 3$ and $\dot{y}_0(0) = -7$ and $x(t) = 0$

we apply MATLAB's `dsolve` command to determine the zero-input response when:

(a) $k = 3$

(b) $k = 4$

(c) $k = 40$

MATLAB code

(a)
```
syms y(t)  % Define y as a symbolic function of t
ode = diff(y, 2) + 4*diff(y, 1) + 3*y == 0;
cond1 = y(0) == 3;
Dy = diff(y, 1); cond2 = Dy(0) == -7;
yo = dsolve(ode, cond1, cond2)
```
[output: `yo = exp(-t) + 2exp(-3*t)`]

(b)
```
syms y(t)  % Define y as a symbolic function of t
ode = diff(y, 2) + 4*diff(y, 1) + 4*y == 0;
cond1 = y(0) == 3;
Dy = diff(y, 1); cond2 = Dy(0) == -7;
yo = dsolve(ode, cond1, cond2)
```
[output: `yo= -exp(-2*t)*(t - 3)`]

(c)
```
syms y(t)
ode = diff(y, 2) + 4*diff(y, 1) + 40*y == 0;
cond1 = y(0) == 3;
Dy = diff(y, 1);
cond2 = Dy(0) == -7;
yo = dsolve(ode, cond1, cond2)
```
[output: `yo = (exp(-2*t)*(18*cos(6*t) - sin(6*t)))/6`]

Outline

- zero-input response
- **impulse response**
- convolution and zero-state response
- system stability

Impulse response

- the (unit) **impulse response**, denoted by $h(t)$, is the output of the system when the input is $x(t) = \delta(t)$ and all initial conditions are zero
- an LTI system is causal if and only if $h(t) = 0$ for $t < 0$

Impulse response for linear differential system

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) y(t) = \left(b_0 D^M + b_1 D^{M-1} + \cdots + b_{M-1} D + b_M\right) x(t)$$

- the impulse response satisfies

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) h(t) = \left(b_0 D^M + b_1 D^{M-1} + \cdots + b_{M-1} D + b_M\right) \delta(t)$$

- with zero initial conditions at $t = 0^-$ ($D^k h(0^-) = 0$)

- input $\delta(t)$ creates nonzero initial conditions (energy storages) at $t = 0^+$
- i.c. creates output consisting of system's characteristic modes for $t \geq 0^+$

Impulse response form

- for $M \leq N$, the impulse response has the form

$$h(t) = b_0\delta(t) + \text{characteristic modes terms}$$

  where b_0 is coefficient of D^N in $P(D)$

- if $M > N$ we can get impulse derivatives at $t = 0$ (impractical case)

Example 3.3 (impulse matching)

find the impulse response $h(t)$ specified by

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

Solution: $b_0 = 0$, so $h(t)$ consists of the characteristic modes only; characteristic polynomial is $\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3)$ and the roots are $-2, -3$

hence, the impulse response $h(t)$ has the form:

$$h(t) = (c_1 e^{-2t} + c_2 e^{-3t})u(t)$$

letting $x(t) = \delta(t)$ and $y(t) = h(t)$ in the differential equation, we obtain

$$\ddot{h}(t) + 5\dot{h}(t) + 6h(t) = \dot{\delta}(t) + \delta(t)$$

we have $h(0^-) = 0$ and $\dot{h}(0^-) = 0$, but the application of an impulse at $t = 0$ creates new initial conditions at $t = 0^+$; let $h\,(0^+) = K_1$ and $\dot{h}\,(0^+) = K_2$

moreover, the jump discontinuities in $h(t)$ and $\dot{h}(t)$ at $t = 0$ creates impulse terms

$$\dot{h}(0) = K_1\delta(t), \qquad \ddot{h}(0) = K_1\dot{\delta}(t) + K_2\delta(t)$$

substituting in the equation and matching the coefficients of impulse terms:

$$K_1 = 1, \quad 5K_1 + K_2 = 1 \quad \Longrightarrow \quad K_1 = 1,\ K_2 = -4$$

so $h\left(0^+\right) = K_1 = 1$ and $\dot{h}\left(0^+\right) = K_2 = -4$

using these initial conditions $h(t) = (c_1e^{-2t} + c_2e^{-3t})u(t)$, we have

$$\begin{aligned} h(0^+) &= c_1 + c_2 = 1 \\ \dot{h}(0^+) &= -2c_1 - 3c_1 = -4 \end{aligned}$$

these two simultaneous equations yield $c_1 = -1$ and $c_2 = 2$; therefore,

$$h(t) = \left(-e^{-2t} + 2e^{-3t}\right) u(t)$$

Simplified impulse matching method

for an LTIC system with $M \leq N$, the unit impulse response $h(t)$ has the form:

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]\, u(t)$$

- b_0 is coefficient of D^N in $P(D)$
- $y_n(t)$ is a linear combination of the characteristic modes of the system with

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) = \cdots = y_n^{(N-2)}(0) = 0 \quad \text{and} \quad y_n^{(N-1)}(0) = 1$$

- for example:

$$N = 1 : y_n(0) = 1$$
$$N = 2 : y_n(0) = 0, \dot{y}_n(0) = 1$$
$$N = 3 : y_n(0) = \dot{y}_n(0) = 0, \ddot{y}_n(0) = 1, \quad \text{...etc}$$

Example 3.4

determine the unit impulse response $h(t)$ for a system specified by the equation

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

Solution: this is a second-order system ($N = 2$) having the characteristic polynomial $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$ and the characteristic roots are $\lambda = -1, \lambda = -2$; thus,

$$y_n(t) = c_1 e^{-t} + c_2 e^{-2t}$$

taking derivative $\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$ and using the initial conditions $\dot{y}_n(0) = 1$ and $y_n(0) = 0$, we have

$$y_n(0) = 0 = c_1 + c_2$$
$$\dot{y}_n(0) = 1 = -c_1 - 2c_2$$

solving gives $c_1 = 1$ and $c_2 = -1$; thus, $y_n(t) = e^{-t} - e^{-2t}$

since $P(D) = D$ and $b_0 = 0$, we have

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]\,u(t) = [Dy_n(t)]\,u(t) = (-e^{-t} + 2e^{-2t})u(t)$$

Using MATLAB to find the impulse response

use MATLAB to determine the impulse response $h(t)$ for the differential equation

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

- a second-order system with $b_0 = 0$
- first we find the zero-input component with $y(0^-) = 0$, and $\dot{y}(0^-) = 1$
- since $P(D) = D$, the zero-input response is differentiated and the impulse response immediately follows as $h(t) = 0\delta(t) + [Dy_n(t)]\, u(t)$

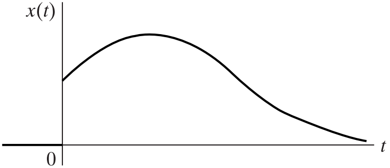
```
syms y(t)
ode = diff(y, 2) + 3*diff(y, 1) + 2*y == 0;
cond1 = y(0) == 0;
Dy = diff(y, 1);
cond2 = Dy(0) == 1;
y_n = dsolve(ode, cond1, cond2);
h = diff(y_n)
```

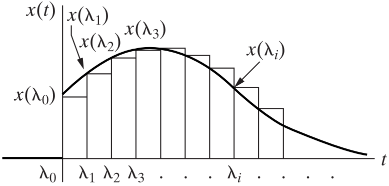
[output: `h=2*exp(-2*t) - exp(-t)`]

Outline

- zero-input response
- impulse response
- **convolution and zero-state response**
- system stability

Derivation of zero-state response of LTI system





we can approximate $x(t)$ by a series of rectangular pulses of uniform width $\Delta\lambda$

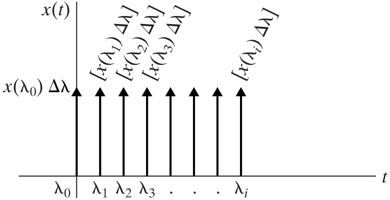
$$x(t) = x_0(t) + x_1(t) + \cdots + x_i(t) + \cdots$$

where $x_i(t)$ is a rectangular pulse with value $x(\lambda_i)$ between λ_i and λ_{i+1}:

$$x_i(t) = x(\lambda_i)\left[u\left(t - \lambda_i\right) - u\left(t - (\lambda_i + \Delta\lambda)\right)\right]$$

the next step in the approximation of $x(t)$ is to make $\Delta\lambda$ small enough that the ith component can be approximated by an impulse function of strength $x\left(\lambda_i\right)\Delta\lambda$

$$\lim_{\Delta\lambda\to 0} \frac{u\left(t - \lambda_i\right) - u\left(t - (\lambda_i + \Delta\lambda)\right)}{\Delta\lambda} = \delta(\lambda - \lambda_i)$$



$$x(t) = x(\lambda_0)\Delta\lambda\delta(t-\lambda_0) + x(\lambda_1)\Delta\lambda\delta(t-\lambda_1) + \cdots + x(\lambda_i)\Delta\lambda\delta(t-\lambda_i) + \cdots$$

using linearity and time-invariance the response function $y(t)$ consists of the sum of a series of uniformly delayed impulse responses

$$\begin{aligned} y(t) &= x(\lambda_0)\,\Delta\lambda h(t-\lambda_0) + x(\lambda_1)\,\Delta\lambda h(t-\lambda_1) + \cdots \\ &= \sum_{i=0}^{\infty} x(\lambda_i)\,h(t-\lambda_i)\,\Delta\lambda \end{aligned}$$

as $\Delta\lambda \rightarrow 0$, the summation approaches a continuous integration, hence

$$\sum_{i=0}^{\infty} x\left(\lambda_i\right) h\left(t-\lambda_i\right) \Delta\lambda \rightarrow \int_0^{\infty} x(\lambda) h(t-\lambda) d\lambda$$

therefore,

$$y(t) = \int_0^{\infty} x(\lambda) h(t-\lambda) d\lambda$$

if $x(t)$ exists over all time, then the lower limit becomes $-\infty$; thus, in general

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda$$

here the $*$ denotes the *convolution* operation read as "$h(t)$ is convolved with $x(t)$ "

Zero-state response of LTI systems

convolution integral of two functions $x_1(t)$ and $x_2(t)$ is

$$x_1(t) * x_2(t) \triangleq \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$$

Zero state response

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- ZSR of an LTIC system is the convolution of input $x(t)$ and impulse response $h(t)$
- if $x(t)$ and $h(t)$ are both causal, then the response $y(t)$ is also causal:

$$y(t) = x(t) * h(t) = \begin{cases} \int_{0-}^{t} x(\tau)h(t-\tau)d\tau & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Example 3.5

for an LTIC system with impulse response $h(t) = e^{-2t}u(t)$, determine response $y(t)$ for input $x(t) = e^{-t}u(t)$

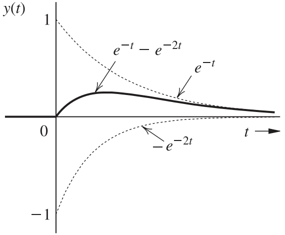
Solution: both $x(t)$ and $h(t)$ are causal, therefore,

$$y(t) = \int_0^t e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau$$

$$= e^{-2t}\int_0^t e^{\tau}d\tau = e^{-2t}(e^t - 1) = e^{-t} - e^{-2t} \quad t \geq 0$$

thus,

$$y(t) = (e^{-t} - e^{-2t})u(t)$$



Convolution properties

Commutative

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

Distributive

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

Associative

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

Shifting: if $x_1(t) * x_2(t) = y(t)$ then

$$x_1(t - t_1) * x_2(t - t_2) = y(t - t_1 - t_2)$$

Convolution with an impulse

$$x(t) * A\delta(t - t_0) = Ax(t - t_0)$$

Differentiation: if $y(t) = x_1(t) * x_2(t)$ then

$$\dot{y}(t) = \dot{x}_1(t) * x_2(t) = x_1(t) * \dot{x}_2(t)$$

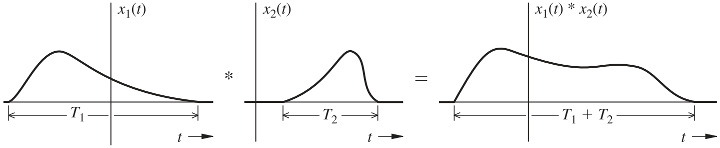
Scaling: if $y(t) = x_1(t) * x_2(t)$ then

$$y(at) = |a|\, x_1(at) * x_2(at)$$

Area: if $y(t) = x_1(t) * x_2(t)$ then

$$\text{area of } y = (\text{area of } x_1) \times (\text{area of } x_2)$$

Width: width of $x_1(t) * x_2(t)$ is $T_1 + T_2$ where T_1, T_2 are widths of $x_1(t), x_2(t)$



Example 3.6 (convolution table)

the convolution of common pairs of functions are already known and can be found from the convolution table

use the convolution table to find the following convolutions:

(a) $e^{-t}u(t) * u(t)$
(b) $e^{-t}u(t) * e^{-t}u(t)$
(c) $e^{-t}u(t) * e^{-2t}u(t)$
(d) $e^{-t}u(t) * \sin(3t)u(t)$

Solution:

(a) $(1 - e^{-t})\, u(t)$
(b) $e^{-t}u(t) * e^{-t}u(t) = te^{-t}u(t)$
(c) $e^{-t}u(t) * e^{-2t}u(t) = \left(e^{-t} - e^{-2t}\right) u(t)$
(d) we use pair 12 (in Table) with $\alpha = 0, \beta = 3, \theta = -90^\circ$ and $\lambda = -1$: this gives $\phi = \tan^{-1}(-3/-1) = -108.4^\circ$ and

$$\begin{aligned}\sin(3t)u(t) * e^{-t}u(t) &= \frac{(\cos 18.4^\circ)e^{-t} - \cos(3t + 18.4^\circ)}{\sqrt{10}} u(t) \\ &= \frac{0.9486e^{-t} - \cos(3t + 18.4^\circ)}{\sqrt{10}} u(t)\end{aligned}$$

Example 3.7

an LTI system has an impulse response $h(t) = 2e^{-3t}u(t)$; determine the (zero-state) response of the system if the input is $x(t) = u(t) - u(t - 1/3)$

Solution: for input $x(t) = u(t) - u(t - 1/3)$, we have

$$\begin{aligned} y(t) = h(t) * x(t) &= h(t) * u(t) - h(t) * u(t - 1/3) \\ &= 2(e^{-3t}u(t) * u(t)) - 2(e^{-3t}u(t) * u(t - 1/3)) \end{aligned}$$

using table and the shift property $x_1(t) * x_2(t - 1/3) = y(t - 1/3)$, we get

$$\begin{aligned} y(t) &= (2/3)(1 - e^{-3t})u(t) - (2/3)(1 - e^{-3(t-1/3)})u(t - 1/3) \\ &= (2/3)\left[(1 - e^{-3t})u(t) - (1 - e^{-3(t-1/3)})u(t - 1/3)\right] \end{aligned}$$

Example 3.8

use the convolution table and the differentiation property to find the zero-state response $y(t) = x(t) * h(t)$ of an LTIC system with $h(t) = \mathrm{rect}(t)$ and $x(t) = \mathrm{rect}(t)$ where $\mathrm{rect}(t) = u(t + \frac{1}{2}) - u(t - \frac{1}{2})$

Solution: from diff. property, we have

$$\begin{aligned} y''(t) = x'(t) * h'(t) &= [\delta(t + \tfrac{1}{2}) - \delta(t - \tfrac{1}{2})] * [\delta(t + \tfrac{1}{2}) - \delta(t - \tfrac{1}{2})] \\ &= \delta(t + 1) - 2\delta(t) + \delta(t - 1) \end{aligned}$$

integrating twice, we get

$$\begin{aligned} y'(t) &= u(t + 1) - 2u(t) + u(t - 1) \\ y(t) &= (t + 1)u(t + 1) - 2tu(t) + (t - 1)u(t - 1) \end{aligned}$$

or alternatively,

$$\begin{aligned} y(t) &= x'(t) * \int_{-\infty}^{t} h(\tau)d\tau \\ &= [\delta(t + \tfrac{1}{2}) - \delta(t - \tfrac{1}{2})] * \left((t + \tfrac{1}{2})[u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2})] + u(t - \tfrac{1}{2})\right) \\ &= (t + 1)u(t + 1) - 2tu(t) + (t - 1)u(t - 1) \end{aligned}$$

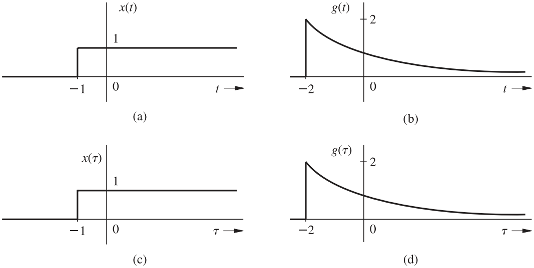
Convolution via graphical procedure

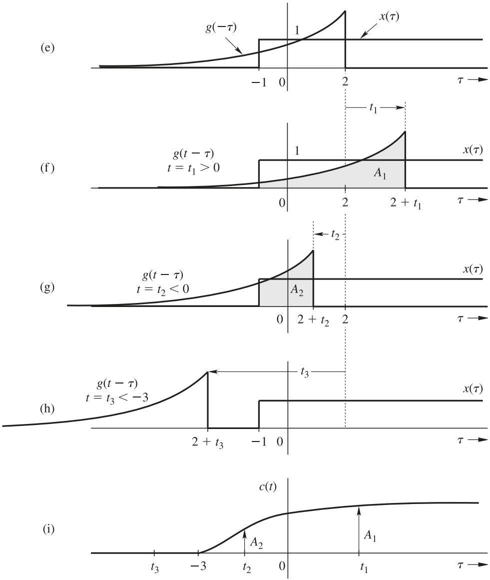
let $c(t)$ be the convolution of $x(t)$ with $g(t)$:

$$c(t) = \int_{-\infty}^{\infty} x(\tau)g(t-\tau)d\tau$$

- integration is performed with respect to τ so that t is treated as constant
- if we know graphs of $x(t)$ and $g(t)$, then we can determine $c(t)$ graphically

Illustration





$c(t_1)$ is the area A_1 and $c(t_2)$ is area A_2

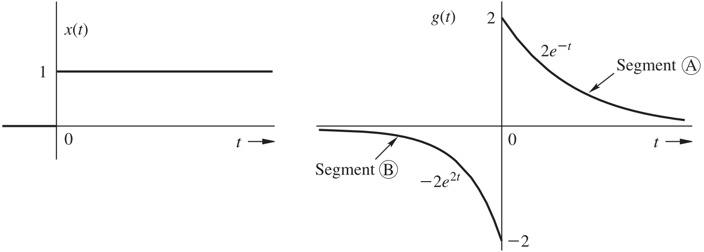
Summary of the graphical procedure

1. keep the function $x(\tau)$ fixed
2. rotate (or invert) $g(\tau)$ about the vertical axis $(\tau = 0)$ to obtain $g(-\tau)$
3. shift $g(-\tau)$ along the τ axis by t_0 seconds to obtain $g(t_0 - \tau)$
4. the area under the product of $x(\tau)$ and $g\,(t_0 - \tau)$ (the shifted frame) is $c\,(t_0)$
5. repeat this procedure, shifting the frame by different values (positive and negative) to obtain $c(t)$ for all values of t

Remark: if the mathematical description of $x(t)$ is simpler than that of $g(t)$, then $g(t) * x(t)$ will be easier to compute than $x(t) * g(t)$

Example 3.9

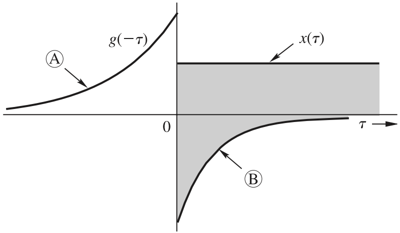
find $c(t) = x(t) * g(t)$ for the signals shown below



Solution: we have $x(t) = 1$ so that $x(\tau) = 1$; notice that

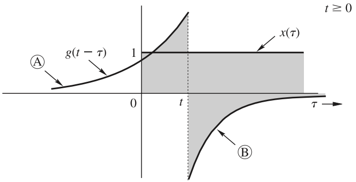
$$g(t) = \begin{cases} 2e^{-t} & \text{segment A} \\ -2e^{2t} & \text{segment B} \end{cases}$$

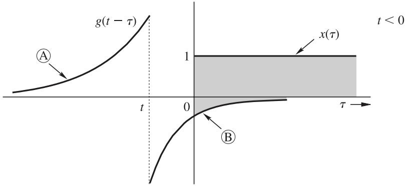
plotting $x(\tau)$ and $g(-\tau)$



we have

$$g(t-\tau) = \begin{cases} 2e^{-(t-\tau)} & \text{segment A} \\ -2e^{2(t-\tau)} & \text{segment B} \end{cases}$$





for $t \geq 0$:

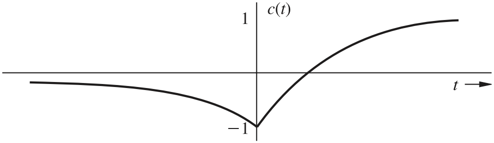
$$c(t) = \int_0^\infty x(\tau)g(t-\tau)d\tau = \int_0^t 2e^{-(t-\tau)}d\tau + \int_t^\infty -2e^{2(t-\tau)}d\tau$$
$$= 2\left(1-e^{-t}\right) - 1 = 1 - 2e^{-t}$$

for $t \leq 0$:

$$c(t) = \int_0^\infty x(\tau)g(t-\tau)d\tau = \int_0^\infty -2e^{2(t-\tau)}d\tau = -e^{2t}$$

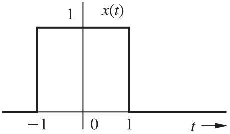
therefore,

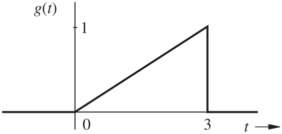
$$c(t) = \begin{cases} 1-2e^{-t} & t \geq 0 \\ -e^{2t} & t \leq 0 \end{cases}$$



Example 3.10

find $x(t) * g(t)$ for the functions $x(t)$ and $g(t)$ shown below



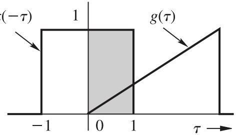


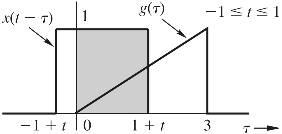
Solution: the signal $x(t)$ has a simpler mathematical description than $g(t)$; hence, we shall determine $g(t) * x(t)$:

$$c(t) = g(t) * x(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$

nonzero segments of $x(t)$ and $g(t)$ are $x(t) = 1$ and $g(t) = \frac{1}{3}t$; hence

$$x(t - \tau) = 1 \quad \text{and} \quad g(\tau) = \frac{1}{3}\tau$$

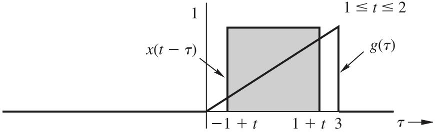




for $-1 \leq t \leq 1$ the two functions overlap over the interval $(0, 1+t)$ so that

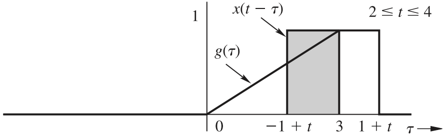
$$c(t) = \int_0^{1+t} g(\tau)x(t-\tau)d\tau = \int_0^{1+t} \frac{1}{3}\tau d\tau = \frac{1}{6}(t+1)^2, \quad -1 \leq t \leq 1$$

$1 \le t \le 2$:



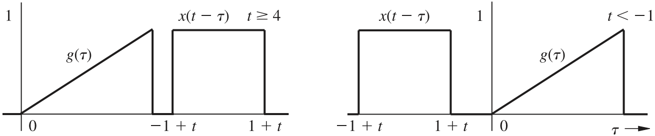
$$c(t) = \int_{-1+t}^{1+t} \frac{1}{3}\tau d\tau = \frac{2}{3}t \quad 1 \le t \le 2$$

$2 \le t \le 4$:



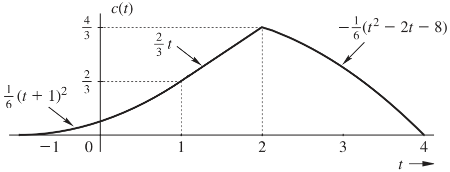
$$c(t) = \int_{-1+t}^{3} \frac{1}{3}\tau d\tau = -\frac{1}{6}\left(t^2 - 2t - 8\right) \quad 2 \le t \le 4$$

$t \geq 4$ and $t < -1$, we have $c(t) = 0$



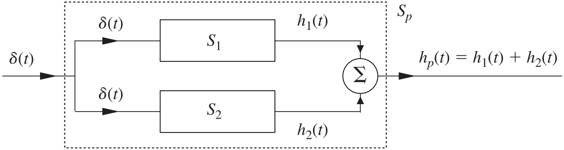
combining our results:

$$c(t) = \begin{cases} \frac{1}{6}(t+1)^2 & -1 \leq t < 1 \\ \frac{2}{3}t & 1 \leq t < 2 \\ -\frac{1}{6}\left(t^2 - 2t - 8\right) & 2 \leq t < 4 \\ 0 & \text{otherwise} \end{cases}$$

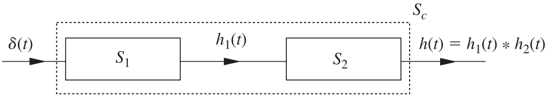


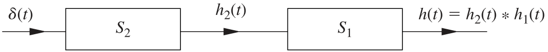
Parallel and cascade systems impulse response

Parallel connection



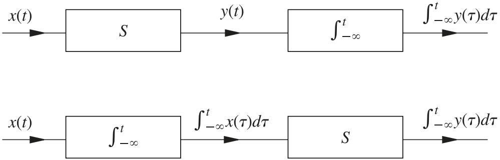
Cascade connection





Cascade systems properties

- using the commutative property of convolution, we have



- this means that $x(t) \Longrightarrow y(t)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \Longrightarrow \int_{-\infty}^{t} y(\tau)d\tau$$

- replacing the integrator with a differentiator, we can show that

$$\frac{dx(t)}{dt} \Longrightarrow \frac{dy(t)}{dt}$$

- the cascade system $h(t)$ with its **inverse system** $h_i(t)$ is an identity system:

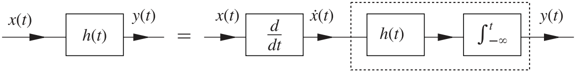
$$h(t) * h_i(t) = \delta(t)$$

Unit-step response

Unit step response: the *unit step response* (output due to step input $u(t)$) of an LTIC system with impulse $h(t)$ is

$$g(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

- using cascade property, we can represent system as:



- impulse response of the dotted box is $g(t)$; thus

$$y(t) = x(t) * h(t) = \dot{x}(t) * g(t)$$

LTI output due to exponential input

an LTIC system response $y(t)$ to an everlasting exponential e^{st} is

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau = H(s)e^{st}$$

- $H(s)$ is the **transfer function** of the system:

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau$$

- input e^{st} gives output $H(s)e^{st}$ of same form; such an input is called the *eigenfunction* (or characteristic function) of the system
- an alternate definition of the transfer function $H(s)$ of an LTIC system is

$$H(s) = \left.\frac{\text{output signal}}{\text{input signal}}\right|_{\text{input = everlasting exponential } e^{st}}$$

Practical significance of transfer function

- transfer function is defined for, and is meaningful to, LTIC systems only
- practical signals can be expressed as a sum of exponentials (or sinusoids)
- for example, a periodic signal $x(t)$ can be expressed as a sum of exponentials as

$$x(t) = \sum_k X_k e^{s_k t}$$

- response $y(t)$ of an LTIC system with transfer function $H(s)$ to this input $x(t)$ is

$$y(t) = \sum_k H(s_k) X_k e^{s_k t}$$

Transfer function of LTI differential system

$$Q(D)y(t) = P(D)x(t)$$

$$Q(D) = D^N + a_1 D^{N-1} + \cdots + a_{N-1}D + a_N$$
$$P(D) = b_0 D^M + b_1 D^{M-1} + \cdots + b_{M-1}D + b_M$$

Transfer function

$$H(s) = \frac{P(s)}{Q(s)}$$

- to see this, we let $x(t) = e^{st}$ use $y(t) = H(s)e^{st}$:

$$H(s)\left[Q(D)e^{st}\right] = P(D)e^{st}$$

- we have $P(D)e^{st} = P(s)e^{st}$ and $Q(D)e^{st} = Q(s)e^{st}$ consequently, $H(s) = P(s)/Q(s)$

Example 3.11 (total response of LTI systems)

$$\text{LTI system total response} = \text{ZIR} + \overbrace{x(t) * h(t)}^{\text{ZSR}}$$

find the total response for the system

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

with input $x(t) = 10e^{-3t}u(t)$ and initial conditions $y(0^-) = 0, \dot{y}(0^-) = -5$

Solution: the zero-input and the impulse response were found in slides 3.7 and 3.22:

$$y_0(t) = (-5e^{-t} + 5e^{-2t})$$
$$h(t) = (2e^{-2t} - e^{-t})u(t)$$

we now use the convolution table to compute the zero-state response:

$$y_{\text{zsr}}(t) = x(t) * h(t) = 10e^{-3t}u(t) * \left[2e^{-2t} - e^{-t}\right] u(t)$$

using the distributive property of the convolution, we obtain

$$\begin{aligned} y_{\text{zsr}}(t) &= 10e^{-3t}u(t) * 2e^{-2t}u(t) - 10e^{-3t}u(t) * e^{-t}u(t) \\ &= 20[e^{-3t}u(t) * e^{-2t}u(t)] - 10[e^{-3t}u(t) * e^{-t}u(t)] \end{aligned}$$

using the table (pair 4) yields

$$\begin{aligned} y(t) &= \tfrac{20}{-3-(-2)}[e^{-3t} - e^{-2t}]u(t) - \tfrac{10}{-3-(-1)}[e^{-3t} - e^{-t}]u(t) \\ &= -20(e^{-3t} - e^{-2t})u(t) + 5(e^{-3t} - e^{-t})u(t) \\ &= (-5e^{-t} + 20e^{-2t} - 15e^{-3t})u(t) \end{aligned}$$

therefore,

$$\text{total response } = \underbrace{(-5e^{-t} + 5e^{-2t})}_{\text{zero-input response}} + \underbrace{(-5e^{-t} + 20e^{-2t} - 15e^{-3t})}_{\text{zero-state response}} \quad t \geq 0$$

Natural and forced response

Natural response: the *natural response* $y_n(t)$ is the the part resulting from the combination of all the characteristic mode terms in the total response

Forced response: the *forced response* $y_\phi(t)$ is the part consisting entirely of noncharacteristic mode terms

- the forced response is the *particular solution* of the differential equation; it is the part of the response the form of which is determined by the input signal
- the natural response is the *homogeneous solution* of the differential equation, where the constants are determined such that the sum of the particular solution and the homogeneous solution satisfies the given initial condition

Example: the total response of the previous RLC example can also be expressed as

$$\text{total current } = \underbrace{\left(-10e^{-t} + 25e^{-2t}\right)}_{\text{natural response } y_n(t)} + \underbrace{\left(-15e^{-3t}\right)}_{\text{forced response } y_\phi(t)} \qquad t \geq 0$$

Outline

- zero-input response
- impulse response
- convolution and zero-state response
- **system stability**

BIBO (external) stability

- system is BIBO stable if every bounded input produces a bounded output
- an LTIC system is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

Examples

- a system with $h(t) = u(t)$ is BIBO unstable since

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{0}^{\infty} d\tau = t\Big|_0^{\infty} = \infty$$

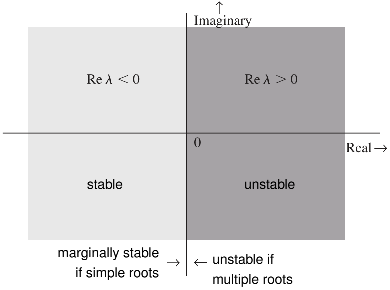
- a system with $h(t) = e^{-t}u(t)$ is BIBO stable since

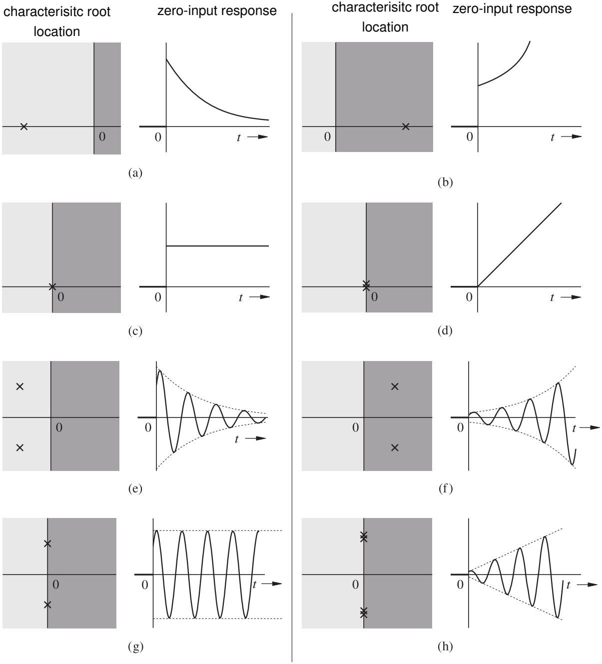
$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{0}^{\infty} e^{-t} d\tau = -e^{-t}\Big|_0^{\infty} = 1$$

Asymptotic (internal) stability

the LTIC differential system described is

1. *asymptotically stable* if, and only if, all the characteristic roots are in the LHP
2. *marginally stable* if, and only if, there are no roots in the RHP, and there are some unrepeated roots on the imaginary axis
3. *unstable* if, and only if, one or both of the following holds:
   - at least one root is in the RHP
   - there are repeated roots on the imaginary axis



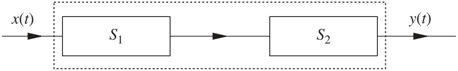

characterisitc root location
zero-input response
characterisitc root location
zero-input response
0
t
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)

- for an LTIC system, if characteristic root λ_k is in the LHP, then the corresponding mode $e^{\lambda_k t}$ is absolutely integrable
- if λ_k is in the RHP or on the imaginary axis, then $e^{\lambda_k t}$ is not absolutely integrable

Relationship between BIBO and asymptotic stability

- an asymptotically (internally) stable system is BIBO-stable
- BIBO unstable implies asymptotically (internally) unstable
- BIBO stability does not imply asymptotic (internal) stability
- marginally stable or asymptotically unstable *LTI system* is BIBO-unstable

Example 3.12



the impulse response of these systems are $h_1(t) = \delta(t) - 2e^{-t}u(t)$ and $h_2(t) = e^t u(t)$; determine the BIBO and asymptotic stability of the system

Solution: the composite system impulse response $h(t)$ is

$$h(t) = h_1(t) * h_2(t) = \left[\delta(t) - 2e^{-t}u(t)\right] * e^t u(t)$$
$$= e^t u(t) - 2\left[\frac{e^t - e^{-t}}{2}\right] u(t) = e^{-t}u(t)$$

- composite system is BIBO-stable because $h(t)$ is absolutely integrable
- subsystem S_2 has a characteristic root 1; hence, S_2 is asymptotically unstable
- so the whole system is unstable
- this shows that BIBO stability does not always imply asymptotic stability

Example 3.13

investigate the asymptotic and the BIBO stability of the LTIC systems:

(a) $(D+1)(D^2+4D+8)y(t) = (D-3)x(t)$
(b) $(D-1)(D^2+4D+8)y(t) = (D+2)x(t)$
(c) $(D+2)(D^2+4)y(t) = (D^2+D+1)x(t)$
(d) $(D+1)(D^2+4)^2y(t) = (D^2+2D+8)x(t)$

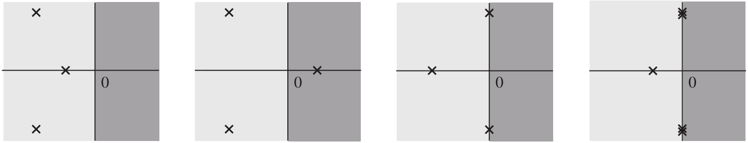
Solution: the characteristic roots of the systems are

(a) $-1, -2 \pm j2$; asymptotically stable (all roots in LHP) and BIBO-stable

(b) $1, -2 \pm j2$; unstable (one root in RHP) and BIBO-unstable

(c) $-2, \pm j2$; marginally stable (unrepeated roots on imaginary axis) and no roots in RHP; BIBO-unstable

(d) $-1, \pm j2, \pm j2$; unstable (repeated roots on imaginary axis); BIBO-unstable



(a) (b) (c) (d)

References

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press.
- M. J. Roberts, *Signals and Systems: Analysis Using Transform Methods and MATLAB*, McGraw Hill.