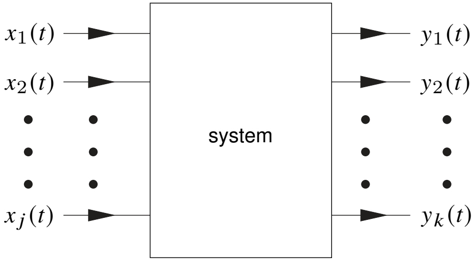
2. Continuous-time systems

- CT systems
- classifications of CT systems
- modeling of basic systems

System

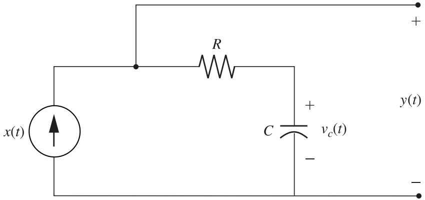
a *system* is an entity that processes input signals to provide output signals

- a system that operates on CT-signals is a *continuous-time systems*
- to excite a system means to apply energy that causes it to respond
- a system can have multiple inputs and multiple outputs (MIMO)



Examples

- *amplifier:* $y(t) = \alpha x(t)$
- *integrator:* $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$
- *RC-circuit*



the input current $x(t)$ and output voltage $y(t)$ are related by:

$$y(t) = Rx(t) + v_C(t_0) + \frac{1}{C} \int_{t_0}^{t} x(\tau) d\tau, \quad t \geq t_0$$

System analysis and design

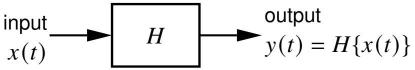
the study of systems consists of three major areas:

- **system modeling:** the mathematical equations relating the outputs to the inputs are called the *system model*
- **system analysis:** how to determine the system outputs for the given inputs and a given mathematical model of the system
- **system design (synthesis):** how to construct a system that will produce a desired set of outputs for the given inputs

Block diagrams

in system analysis it is common and useful to represent systems by *block diagrams*

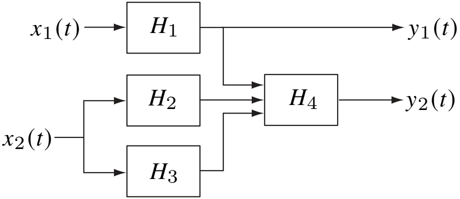
Single-input single-output



- input $x(t)$ is operated on by operator H to produce the output signal $y(t)$
- the operator H could perform just about any operation imaginable

Interconnected systems

a system is often described and analyzed as an assembly of **components**

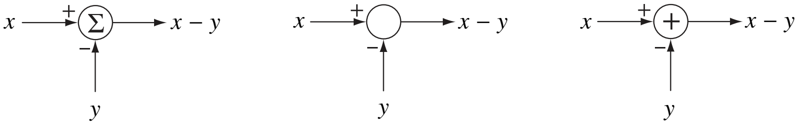


- a component is a smaller, simpler system
  - to a circuit designer, components are resistors, capacitors, inductors, operational amplifiers and so on, and systems are power amplifiers, A/D converters, modulators, filters and so forth
  - to an automobile designer components are wheels, engines, bumpers, lights, seats and the system is the automobile
- by knowing the mathematical model of the components, an engineer can predict the behavior (output) of the system

Common block diagram operations

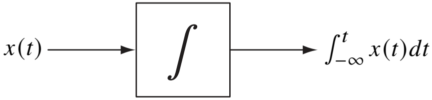
Amplifier (scalar multiplication)

Summation (addition)

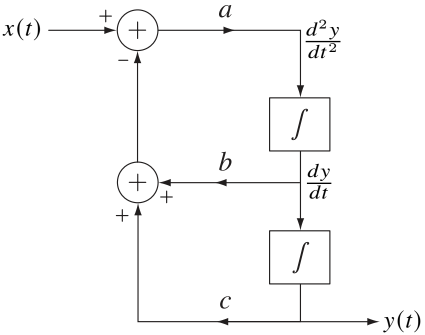


(a) (b) (c)

Integrator



Example 2.1



$$\frac{dy^2}{dt^2} = a\Big(x(t) - [b\frac{dy}{dt} + cy(t)]\Big)$$

or

$$\frac{dy^2}{dt^2} + (ab)\frac{dy}{dt} + (ac)y(t) = ax(t)$$

Outline

- CT systems
- **classifications of CT systems**
- modeling of basic systems

Linear systems

a system H is

- *homogeneous* if $x \to y$, then $\alpha x \to \alpha y$ for any number α
- *additive* if $x_1 \to y_1$, and $x_2 \to y_2$, then $x_1 + x_2 \to y_1 + y_2$

Linear systems: a system is *linear* if it is both homogeneous and additive

$$x_1 \longrightarrow y_1$$
$$x_2 \longrightarrow y_2$$

then for any numbers α_1, α_2

$$\alpha_1 x_1 + \alpha_2 x_2 \longrightarrow \alpha_1 y_1 + \alpha_2 y_2$$

the above is called the *superposition property*

Example 2.2

determine whether the following systems are linear or nonlinear

(a) $\dfrac{dy(t)}{dt} + 3y(t) = x(t)$

(b) $y(t)\dfrac{dy(t)}{dt} + 3y(t) = x(t)$

(c) $y(t) = e^{x(t)}$

Solution:

(a) let $y_1(t)$ and $y_2(t)$ to be the outputs for inputs $x_1(t)$ and $x_2(t)$; then,

$$\frac{dy_1(t)}{dt} + 3y_1(t) = x_1(t) \qquad \frac{dy_2(t)}{dt} + 3y_2(t) = x_2(t)$$

multiplying the first equation by α_1 and the second by α_2 and adding, gives

$$\frac{d}{dt}[\alpha_1 y_1(t) + \alpha_2 y_2(t)] + 3[\alpha_1 y_1(t) + \alpha_2 y_2(t)] = \alpha_1 x_1(t) + \alpha_2 x_2(t),$$

which is the system equation with

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t), \qquad y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

hence, superposition is satisfied and the system is linear

(b) if $x(t) \longrightarrow y(t)$, then we have

$$y(t)\frac{dy(t)}{dt} + 3y(t) = x(t)$$

multiplying by α, we have

$$\alpha y(t)\frac{dy(t)}{dt} + 3\alpha y(t) = \alpha x(t),$$

which is not equal to

$$\alpha y(t)\frac{d[\alpha y(t)]}{dt} + 3\alpha y(t) = \alpha x(t)$$

hence, the system is nonlinear

(c) for input $\alpha x(t)$, we have $y(t) = e^{\alpha x(t)} \neq \alpha y(t)$

Total response of a linear system

Decomposition property of linear systems

total response = zero-input response + zero-state response

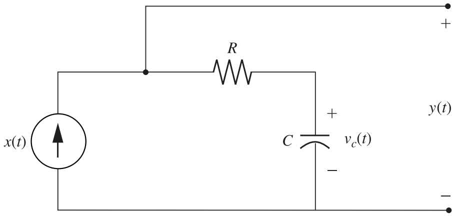
Zero-input response (ZIR)

- ZIR is the output that results only from initial conditions at $t = 0$
- with zero input $x(t) = 0$ for $t \geq 0$

Zero-state response (ZSR)

- ZSR is the output that results from input $x(t)$ for $t \geq 0$
- with zero initial conditions
- when all the initial conditions are zero, the system is said to be in *zero state*

Example: for the circuit in slide 2.3 (with $t_0 = 0$)



we have

$$y(t) = = \underbrace{v_C(t_0)}_{\text{ZIR}} + \underbrace{Rx(t) + \frac{1}{C}\int_{t_0}^{t} x(\tau)d\tau,}_{\text{ZSR}} \quad t \geq 0$$

Linearity implication

if we can write $x(t)$ as

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) + \cdots + \alpha_m x_m(t)$$

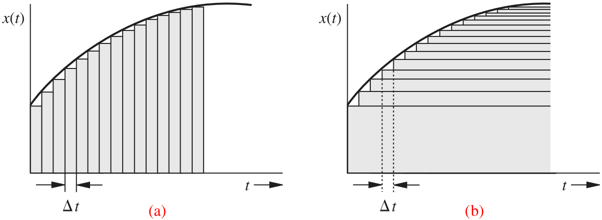
then if the system is linear, the output is

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) + \cdots + \alpha_m y_m(t)$$

- $y_k(t)$ is the zero-state response to input $x_k(t)$
- we can find $y(t)$ by finding responses $y_k(t)$ to the “simpler” components $x_k(t)$

Linearity implication

any signal can be approximated by a sum of rectangular pulses or step-functions



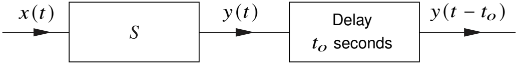
if we know the system response to a unit impulse or unit step input, we can compute the system response to any arbitrary input

Time-invariant systems

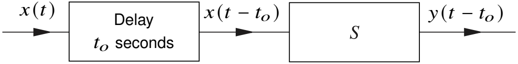
a system is *time invariant* if for input-output $x(t) \rightarrow y(t)$, we have

$$x(t - t_o) \rightarrow y(t - t_o)$$

for any arbitrary t_o (assuming initial conditions are also delayed by t_o)



(a)



(b)

- a system is *time-varying* if the the above does not hold
- CT system that is linear and time-invariant is called *linear-time-invariant continuous system* (LTIC)

Example 2.3

determine the time invariance of the following systems

(a) $y(t) = x(t)u(t)$

(b) $y(t) = \dfrac{d}{dt}x(t)$

(c) $y(t) = e^{-t}x(t)$

(d) $y(t) = e^{x(t)}$

Solution:

(a) input is modified by a time-dependent function $u(t)$ so the system is time-varying; we can show this through a counterexample:

$$x_1(t) = \delta(t+1) \quad \Longrightarrow \quad y_1(t) = 0$$
$$x_2(t) = x_1(t-2) = \delta(t-1) \quad \Longrightarrow \quad y_2(t) = \delta(t-1)$$

since $y_2(t) \neq y_1(t-2) = 0$, the system is time-varying

(b) for input $x(t-t_o)$, we have output

$$y(t-t_o) = \frac{d}{d(t-t_o)}x(t-t_o) = \frac{d}{dt}x(t-t_o),$$

which is the output to a delayed input $x(t-t_o)$; hence, the system is time invariant

(c) the output with delayed input is $e^{-t}x(t-t_o)$, which is not equal to the delayed output $e^{-(t-t_o)}x(t-t_o)$; hence, system is time-varying

(d) for input $x(t-t_o)$, output is $e^{x(t-t_o)} = y(t-t_o)$; hence system is time invariant

Instantaneous and dynamic systems

Instantaneous (memoryless, static) system

- output at any time t depends only on its input(s) at the same time t
- does not depend on any past or future values of the input(s)

Dynamic systems (with memory)

- output depends on future or past values of input(s)
- a *finite-memory system with a memory* T is a system whose output at t depends only on the input signals over the past T seconds (from $t - T$ to t)

Example 2.4

determine whether the following systems are memoryless:

(a) $y(t-1) = 2x(t-1)$

(b) $y(t) = \frac{d}{dt}x(t)$

(c) $y(t) = (t-1)x(t)$

Solution:

(a) memoryless since the output at any time depends on the input at the same time

(b) using the derivative definition

$$y(t) = \lim_{T \to 0} \frac{x(t) - x(t-T)}{T}$$

not memoryless since the output at t depends on more than just the input at t

(c) memoryless since the output at t depends only on the input at the same time

Causal and noncausal systems

Causal systems

- output at t_0 depends only on the input $x(t)$ for $t \leq t_0$
- output does not depend on future input

Noncausal systems

- system that violates the condition of causality (*i.e.*, output depend on future input)
- unrealizable in *real time* but can be realizable with time delay; for example, we can prerecord data; in such cases, the input's future values are available to us

Example 2.5

determine whether the following systems are causal

(a) $y(t) = x(-t)$

(b) $y(t) = x(t + 1)$

(c) $y(t + 1) = x(t)$

Solution:

(a) output at $t = -1$, $y(-1) = x(1)$ depends on future input; hence not causal

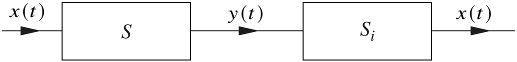
(b) output at time t depends on input at future $t + 1$; thus, the system is not causal

(c) output at time $t + 1$ depends only on past input; hence, causal

Invertible and noninvertible systems

Invertible systems: a system is *invertible* if we can find the input $x(t)$ from the corresponding output $y(t)$

- system that achieves the inverse operation is the *inverse system* for S



- every input have a unique output (one-to-one mapping between input and output)

Noninvertible systems

- a system is *noninvertible* when it is impossible to obtain the input from the output (several different inputs result in the same output)
- examples: two inputs give same output
  - rectifier: $y(t) = |x(t)|$
  - $y(t) = \sin(x(t))$

Example 2.6

determine whether the following systems are invertible

(a) $y(t) = x(-t)$

(b) $y(t) = tx(t)$

(c) $y(t) = \frac{d}{dt}x(t)$

(d) $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

Solution:

(a) since $x(t) = y(-t)$ for all t, the system is invertible

(b) we have $x(t) = \frac{1}{t}y(t)$ for all t except $t = 0$; system is noninvertible since we cannot recover $x(0)$

(c) since the derivative of constants are equal, the system is noninvertible; for example, both $x_1(t) = t + 1$ and $x_2(t) = t - 5$ give the same output

(d) invertible because the input can be obtained by taking the derivative of the output; hence, the inverse system equation is $y(t) = dx/dt$

BIBO stable systems

a system is *bounded-input-bounded-output (BIBO) stable* (*externally stable*) if every bounded input results in a bounded output

Example: determine whether the following systems are BIBO-stable

(a) $y(t) = x^2(t)$

(b) $y(t) = tx(t)$

(c) $y(t) = \frac{d}{dt}x(t)$

Solution:

(a) system $y(t) = x^2(t)$ is BIBO stable: if the input is bounded $|x(t)| \leq M_x < \infty$, then $|y(t)| = |x^2(t)| = |x(t)|^2 \leq M_x^2 < \infty$

(b) the bounded-amplitude input $x(t) = u(t)$ produces the output $y(t) = tu(t)$, which grows to infinity as $t \longrightarrow \infty$; thus system is a BIBO-unstable system

(c) the bounded-amplitude input $x(t) = u(t)$ produces the output $y(t) = \delta(t)$ whose amplitude is infinite at $t = 0$; thus, the system is a BIBO-unstable

Linear differential system

$$a_0 \frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_N y(t)$$
$$= b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt} + \cdots + b_M x(t)$$

- order is highest derivative of output N
- the system described by differential equation of the above form is linear
- the system is time-invariant if a_i, b_i are constants (independent of time)
- many practical systems can be modeled by linear differential equations
- we assume that $a_0 = 1$ (if not, then we can always divide both sides by a_0)

Differentiation notations

- the are several notation for differentiation:

$$\dot{y}(t) = y'(t) := \frac{dy(t)}{dt}, \quad \ddot{y}(t) = y''(t) := \frac{d^2y(t)}{dt^2}, \quad \ldots, \quad y^{(N)} := \frac{d^N y(t)}{dt^N}$$

- for convenience, we often use D instead of d/dt:

$$\frac{dy(t)}{dt} := Dy(t), \quad \frac{d^2y(t)}{dt^2} := D^2y(t), \quad \ldots, \quad \frac{d^N y(t)}{dt^N} := D^N y(t)$$

- using the above, the linear differential system becomes

$$(a_0D^N + a_1D^{N-1} + \cdots + a_N)y(t) = (b_0D^M + b_1D^{M-1} + \cdots + b_M)x(t)$$

Integration operation

$$\int_{-\infty}^{t} y(\tau)d\tau := \frac{1}{D}y(t)$$

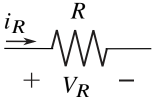
Outline

- CT systems
- classifications of CT systems
- **modeling of basic systems**

Basic electrical elements laws

Resistor

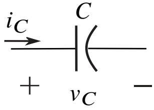
$$v_R = i_R R$$



Capacitor

$$i_C = C\frac{dv_C}{dt}$$

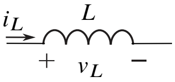
$$v_C(t) = \frac{1}{C}\int_{t_0}^{t} i_C \, d\tau + v_C(t_0)$$



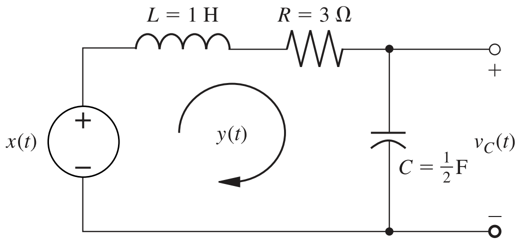
Inductor

$$v_L = L\frac{di_L}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{t_0}^{t} v_L \, d\tau + i(t_0)$$



Example 2.7



find the input-output equation relating the input voltage $x(t)$ to the output current (loop current) $y(t)$

Solution: KVL, gives

$$v_L(t) + v_R(t) + v_C(t) = x(t)$$

using voltage current-law for each element we obtain:

$$\frac{dy(t)}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

differentiating both sides, we get the input-output relation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

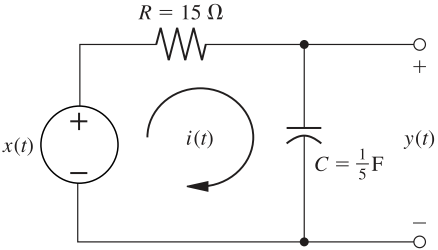
we can write the above as

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

if the inductor voltage $v_L(t)$ is taken as the output, then

$$(D^2 + 3D + 2)v_L(t) = D^2x(t)$$

Example 2.8



find the equation relating input-output if the input is the voltage $x(t)$ and output is

(a) the loop current $i(t)$

(b) the capacitor voltage $y(t)$

Solution:

(a) the loop equation is

$$15i(t) + 5\int_{-\infty}^{t} i(\tau)d\tau = x(t)$$

in operator notation, we have

$$15i(t) + \frac{5}{D}i(t) = x(t)$$

multiplying both sides by D (*i.e.*, differentiating the equation), we obtain

$$(15D + 5)i(t) = Dx(t)$$

(b) using $i(t) = C\dfrac{dy(t)}{dt} = \frac{1}{5}Dy(t)$, we get

$$(3D + 1)y(t) = x(t)$$

if the capacitor voltage $v_C(t)$ is taken as the output, then

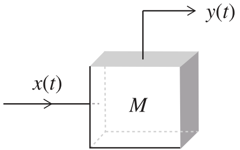
$$(D^2 + 3D + 2)v_C(t) = 2x(t)$$

Mechanical translational laws

the basic elements used in modeling translational systems (moving along a straight line) are ideal masses, linear springs, and dashpots providing viscous damping

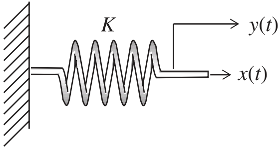
Newton's law of motion: a force $x(t)$ on *mass* M causes a motion $y(t)$ and acceleration $\ddot{y}(t)$

$$x(t) = M\ddot{y}(t) = MD^2y(t)$$



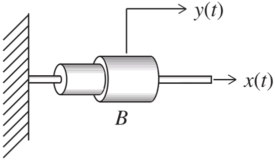
Linear spring: force $x(t)$ required to stretch (or compress) a linear spring with *stiffness* K by amount $y(t)$

$$x(t) = Ky(t)$$



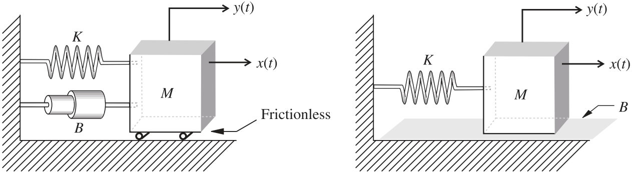
Linear dashpot: the force $x(t)$ moving the dashpot with *damping coefficient* B is proportional to the relative velocity $\dot{y}(t)$ of one surface with respect to the other

$$x(t) = B\dot{y}(t) = BDy(t)$$

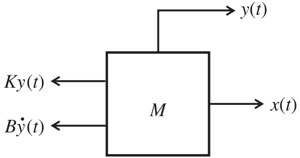


Example 2.9

find the input-output relationship for the translational mechanical system shown below; the input is the force $x(t)$, and the output is the mass position $y(t)$



Solution: in mechanical systems it is helpful to draw a free-body diagram of each junction, which is a point at which two or more elements are connected



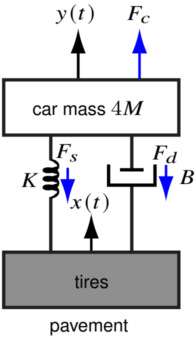
from Newton's second law, the net force must be

$$M\ddot{y}(t) = -B\dot{y}(t) - Ky(t) + x(t)$$

or

$$(MD^2 + BD + K)y(t) = x(t)$$

Example 2.10 (car suspension system)



- input $x(t)$ is vertical displacement of pavement (relative to ground level)
- output $y(t)$ is vertical displacement of the car chassis from its equilibrium position
- M is one-fourth of the car's mass, because the car has four wheels

- forces exerted by the spring F_s and shock absorber F_d depend on the relative displacement $(y - x)$ of the car relative to the pavement
- when $(y - x)$ is positive (car mass moving away from the pavement), the spring force F_s is directed downward; hence, $F_s = -K(y - x)$
- similarly, $F_d = -B\frac{d}{dt}(y - x)$
- using Newton's law, $F_c = Ma = M\frac{d^2y}{dt^2}$, the force equation is $F_c = F_s + F_d$ or

$$M\frac{d^2y}{dt^2} = -K(y - x) - B\frac{d}{dt}(y - x)$$

which can be written as

$$\frac{d^2y}{dt^2} + \frac{B}{M}\frac{dy}{dt} + \frac{K}{M}y = \frac{B}{M}\frac{dx}{dt} + \frac{K}{M}x$$

this is a second-order linear differential system

References

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press.
- M. J. Roberts, *Signals and Systems: Analysis Using Transform Methods and MATLAB*, McGraw Hill.