1. Continuous-time signals

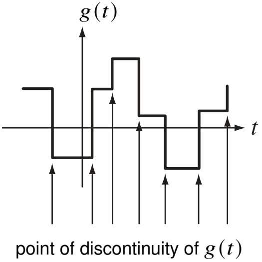
- continuous-time signals
- signal operations
- useful CT signals
- even and odd signals
- signal energy and power

Continuous-time signal

a *continuous-time (CT) signal* is a function $x(t)$ defined at every time t

- voltage, current, audio signals
- light intensity variations in an optical fiber
- position or velocity of moving object

a continuous-time function is not the same as *continuous function*



Sinusoids and exponentials

Sinusoids

$$x(t) = A\cos(2\pi f t + \theta)$$

- f is the (cyclic) *frequency* (in Hertz); $T = 1/f$ is the *period*
- A is the *amplitude* and θ is the *phase* (in degrees or radians)
- $\omega = 2\pi f = 2\pi/T$ is the *radian frequency*
- other form: $A\sin(\omega t + \theta) = A\cos(\omega t + \theta - \pi/2)$

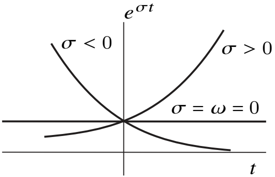
Exponentials

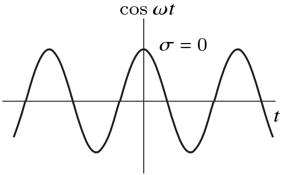
$$\begin{aligned} x(t) = Ae^{st} &= Ae^{(\sigma + j\omega)t} \\ &= Ae^{\sigma t}\left(\cos\omega t + j\sin\omega t\right) \end{aligned}$$

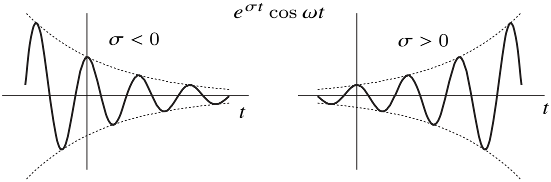
- $s = \sigma + j\omega$ is called *complex frequency*
- $|\omega|$ is called *radian frequency* or frequency of oscillation
- σ is the *decay rate* or *neper frequency*

Signals in terms of exponentials

- constant: $k = ke^{0t}$ ($s = 0$)
- monotonic exponential: $e^{\sigma t}$ ($\omega = 0$)
- sinusoid: $\cos \omega t = \mathrm{Re}(e^{\pm j\omega t})$ ($\sigma = 0,\ \omega = \pm j\omega$)
- exponentially varying sinusoid: $e^{\sigma t} \cos \omega t$ ($s = \sigma \pm j\omega$)







Adding sinusoids with same frequency

the **phasor** of the sinusoid $A\cos(\omega t + \theta)$ is the complex number $Ae^{j\theta} = A\underline{/\theta}$

Adding sinusoids

- adding sinusoids with the *same* frequency can be done via trigonometric identities or *phasors*

$$A_1\cos(\omega t + \theta_1) + A_2\cos(\omega t + \theta_2) = A\cos(\omega t + \theta)$$

- A and θ can computed by using phasors:

$$A_1 e^{j\theta_1} + A_2 e^{j\theta_2} = Ae^{j\theta}$$

Example: find $\cos(\omega t + 60^o) + 5\cos(\omega t - 30^o)$

- we have $e^{j60^o} + 5e^{-j30^o} = 5.099e^{-j18.69^o}$
- therefore,

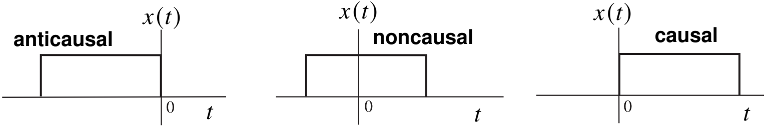
$$\cos(\omega t + 60^o) + 5\cos(\omega t - 30^o) = 5.099\cos(\omega t - 18.69^o)$$

Causal signals

a signal $x(t)$ is *causal* if

$$x(t) = 0 \quad \text{for } t < 0$$

- causal signals do not start before $t = 0$
- a signal $x(t)$ is *anticausal* if $x(t) = 0,\ t \geq 0$
- a signal that has value before and after $t = 0$ is called *noncausal*
- a signal that exists over $-\infty < t < \infty$ is called *everlasting signal*



Periodic and aperiodic signals

a signal $x(t)$ is *periodic* if for some positive constant T

$$x(t) = x(t+T) \quad \text{for all } t$$

- smallest T is called *(fundamental) period* of $x(t)$, denoted by T_0
- $f_0 = 1/T_0$ is *cyclic frequency*; $\omega_0 = 2\pi f_0$ is *radian frequency*
- a periodic signal must be an everlasting signal
- property: areas under $x(t)$ over any interval of duration T_0 are equal

$$\int_a^{a+T_0} x(t)\, dt = \int_b^{b+T_0} x(t)\, dt \triangleq \int_{T_0} x(t)\, dt$$

- a signal is *aperiodic* if it is not periodic

Sum of periodic signals

$$x(t) = x_1(t) + x_2(t)$$

- $x_1(t)$ and $x_2(t)$ are is periodic with periods T_{01} and T_{02}
- $x(t)$ is periodic with period T if T is an integer multiple of T_{01} and T_{02}:

$$qT_{01} = pT_{02} \quad \text{for integers } p, q$$

Fundamental period of $x(t)$ is the *least common multiple* (LCM) of T_{01}, T_{02}

$$T_0 = \text{LCM}(T_{01}, T_{02})$$

- if T_{01}/T_{02} is a rational number, then $x(t)$ is periodic; otherwise, it is aperiodic
- if $T_{01}/T_{02} = p_0/q_0$ for some integers p_0 and q_0 in **smallest form**, then

$$T_0 = \text{LCM}(T_{01}, T_{02}) = q_0 T_{01} = p_0 T_{02}$$

Example 1.1

- signal $x(t) = 3 + t^2$ is aperiodic
- signal $x(t) = e^{-j60\pi t}$ can be expressed as a sum of two periodic signals

$$x(t) = \cos(60\pi t) - j\sin(60\pi t)$$

with the same fundamental period $T_{01} = T_{02} = 2\pi/60\pi = \frac{1}{30}$; thus, $T_0 = \frac{1}{30}$ s

- the signal $x(t) = 10\sin(12\pi t) + 4\cos(18\pi t)$ is the sum of two periodic functions with $T_{01} = 1/6$ second and $T_{02} = 1/9$ second; we have

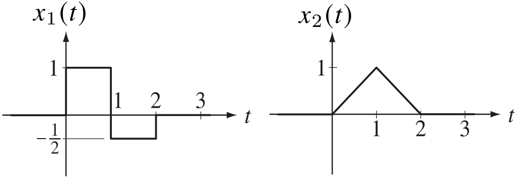
$$T_{01}/T_{02} = \frac{9}{6} = \frac{3}{2} \Rightarrow T_0 = \text{LCM}(\frac{1}{6}, \frac{1}{9}) = 2T_{01} = 3T_{02} = 1/3$$

- the signal $x(t) = 10\sin(12\pi t) + 4\cos(18t)$ is the sum of two periodic functions with $T_{01} = 1/6$ second and $T_{02} = \pi/9$ seconds; the ratio $T_{01}/T_{02} = 2\pi/3$ is irrational; therefore $x(t)$ is aperiodic

Piecewise signals

a *piecewise signal* is a function with different expressions over different intervals

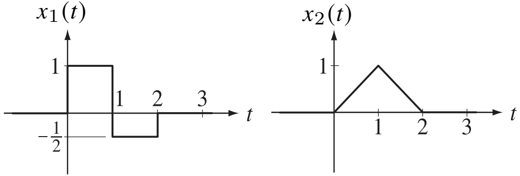
Example



$$x_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1/2 & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad x_2(t) = \begin{cases} t & 0 \le t \le 1 \\ -(t-2) & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

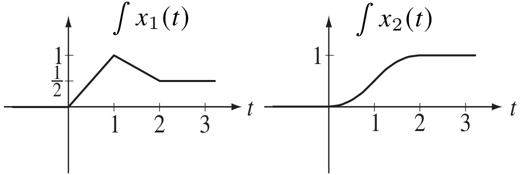
Example 1.2

find and sketch $\int_{-\infty}^{t} x_1(\tau)d\tau$ and $\int_{-\infty}^{t} x_2(\tau)d\tau$ for the signals $x_1(t)$ and $x_2(t)$



Solution:

$$\int_{-\infty}^{t} x_1(\tau)d\tau = \begin{cases} t & 0 \le t \le 1 \\ \frac{3}{2} - \frac{1}{2}t & 1 \le t \le 2 \\ 1/2 & t \ge 2 \end{cases}, \qquad \int_{-\infty}^{t} x_2(\tau)d\tau = \begin{cases} t^2/2 & 0 \le t \le 1 \\ -(\frac{t^2}{2} - 2t + 1) & 1 \le t \le 2 \\ 1 & t \ge 2 \end{cases}$$



Outline

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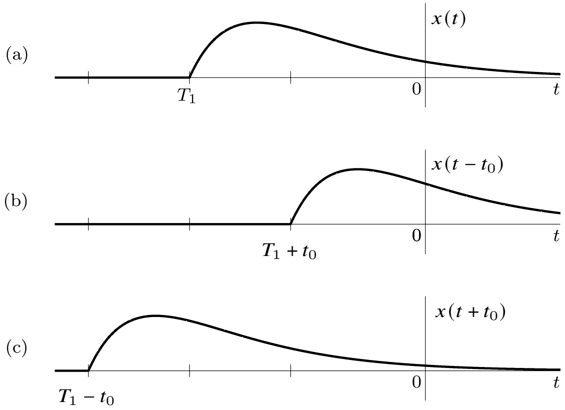
Time shifting

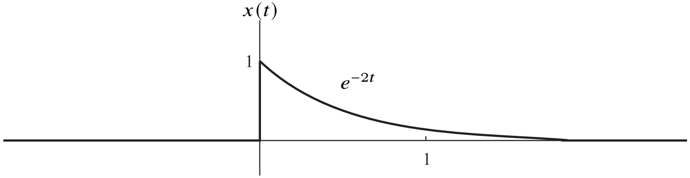
signals can be shifted to the right or left by $t_0 > 0$ seconds:

$x(t - t_0)$ right-shifted (delayed) signal

$x(t + t_0)$ left-shifted (advanced) signal



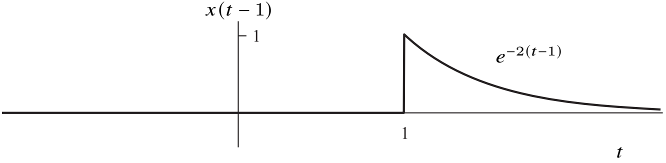
Example 1.3

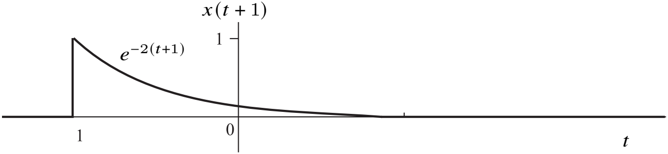


$$x(t) = \begin{cases} e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

sketch and give and expression of $x(t)$ delayed by 1 and advanced by 1

Solution:





$$x(t-1) = \begin{cases} e^{-2(t-1)} & t-1 \geq 0 \\ 0 & t-1 < 0 \end{cases}$$

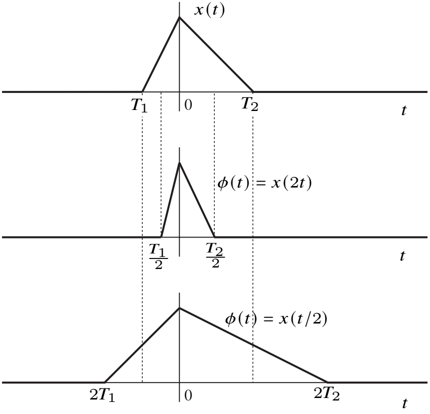
$$x(t+1) = \begin{cases} e^{-2(t+1)} & t+1 \geq 0 \\ 0 & t+1 < 0 \end{cases}$$

Time scaling

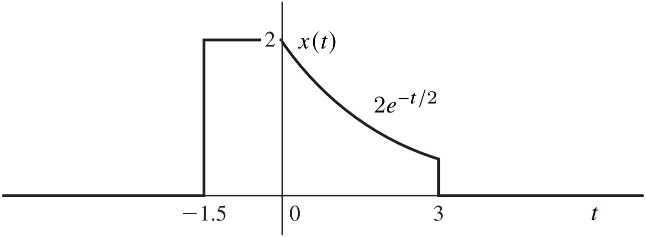
time scaling is the compression or expansion of a signal in time:

$$\phi(t) = x(\alpha t) \quad \text{compression by factor } \alpha > 1$$

$$\phi(t) = x(t/\alpha) \quad \text{expansion by factor } \alpha > 1$$



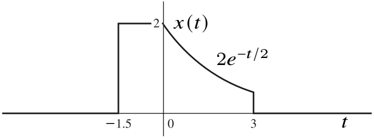
Example 1.4



sketch and give an expression for $x(t)$ time-compressed by factor 3 and time-expanded by factor 2

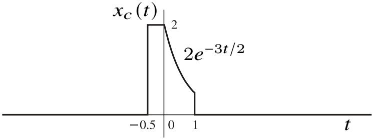
Solution:

$$x(t) = \begin{cases} 2 & -1.5 \leq t < 0 \\ 2e^{-t/2} & 0 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}$$



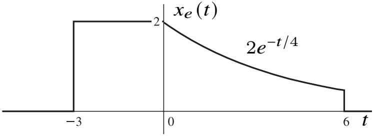
compressed signal

$$x_c(t) = x(3t) = \begin{cases} 2 & -1.5 \leq 3t < 0 \\ 2e^{-3t/2} & 0 \leq 3t < 3 \\ 0 & \text{otherwise} \end{cases}$$



expanded signal

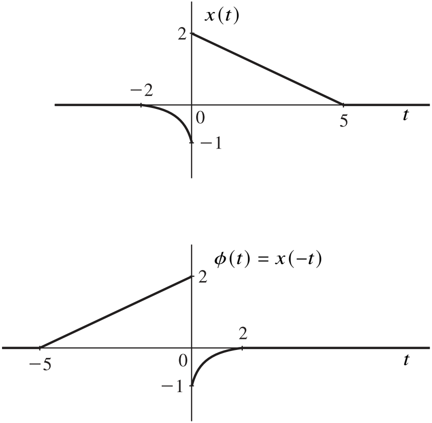
$$x_e(t) = x(t/2) = \begin{cases} 2 & -1.5 \leq t/2 < 0 \\ 2e^{-t/4} & 0 \leq t/2 < 3 \\ 0 & \text{otherwise} \end{cases}$$



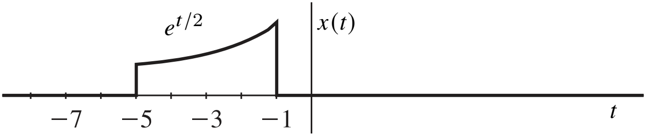
Time reversal

time-reversal is the reflection about the *vertical* axis

$$\phi(t) = x(-t)$$

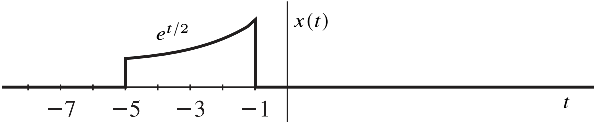


Example 1.5



sketch and mathematically describe $x(-t)$

Solution:



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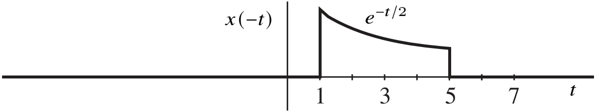


$$x(t) = \begin{cases} e^{t/2} & -5 < t \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

$$x(-t) = \begin{cases} e^{-t/2} & -5 < -t \leq -1 \ (1 \leq t < 5) \\ 0 & \text{otherwise} \end{cases}$$

Combined operations

$$x(\alpha t - t_0) = x\left(\alpha(t - \tfrac{t_0}{\alpha})\right)$$

1. time shift, then time scale the shifted signal

$$x(t) \overset{\text{time shift by } t_0}{\Longrightarrow} x(t - t_0) \overset{\text{time scale by } \alpha}{\Longrightarrow} x(\alpha t - t_0)$$

2. time scale, then time shift

$$x(t) \overset{\text{time scale by } \alpha}{\Longrightarrow} x(\alpha t) \overset{\text{time shift by } t_0/\alpha}{\Longrightarrow} x(\alpha t - t_0)$$

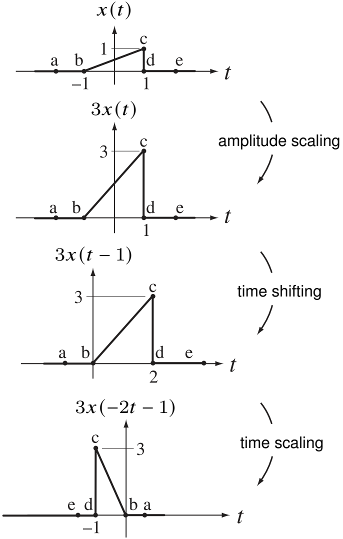
Other form

$$x\left(\frac{t - t_0}{\alpha}\right)$$

$$x(t) \overset{\text{time scale by } 1/\alpha}{\Longrightarrow} x(t/\alpha) \overset{\text{time shift by } t_0}{\Longrightarrow} x\left(\frac{t - t_0}{\alpha}\right)$$

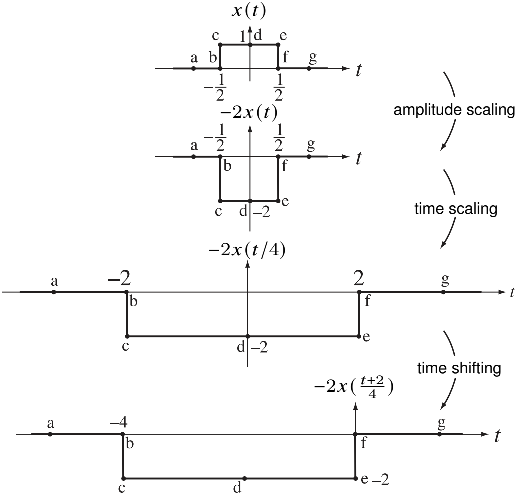
Example 1.6

sketch $3x(-2t-1)$ from $x(t)$



Example 1.7

sketch $-2x(\frac{t+2}{4})$ from $x(t)$



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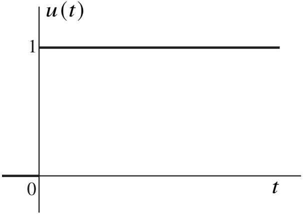
Unit step

unit step function:

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \le 0 \end{cases}$$



- $u(t)$ is sometimes defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ 0.5 & t = 0 \\ 0 & t < 0 \end{cases}$$

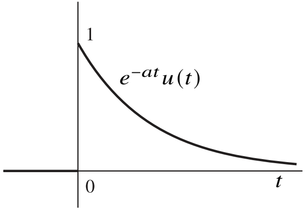
  which is convenient for theoretical purposes

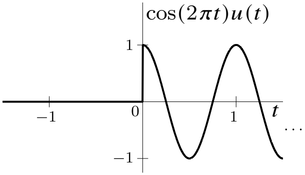
- for real-world signals applications however, it makes no practical difference

Unit-step and causal signals

unit-step is useful to describe causal signals

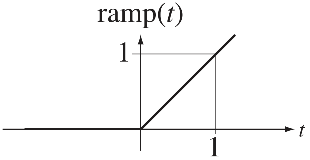
- $e^{-at}u(t)$ is zero for $t < 0$ and e^{-at} for $t \geq 0$
- similarly for $\cos(2\pi t)u(t)$





Unit ramp

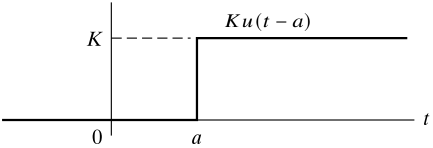
$$\mathrm{ramp}(t) = tu(t) = \begin{cases} t & t > 0 \\ 0 & t \leq 0 \end{cases}$$
$$= \int_{-\infty}^{t} u(\tau)d\tau$$



Shifting and reversal of unit step

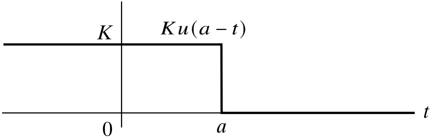
Shifted step: a step signal equal to K that occurs at $t = a$ is expressed as

$$Ku(t-a) = \begin{cases} 0 & t < a \\ K & t > a \end{cases}$$



Shifted and reversed step: a step signal equal to K for $t < a$ is written as

$$Ku(a-t) = \begin{cases} K & t < a \\ 0 & t > a \end{cases}$$

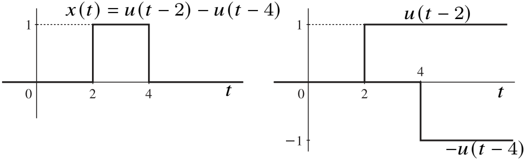


Rectangular pulse

a *rectangular pulse* from t_1 to t_2 can be represented as $u(t - t_1) - u(t - t_2)$

Examples

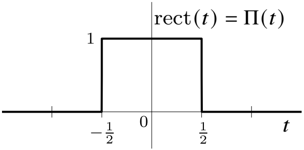
- rectangular pulse from 2 to 4



(a) (b)

- the *unit rectangle* (unit gate) is defined as

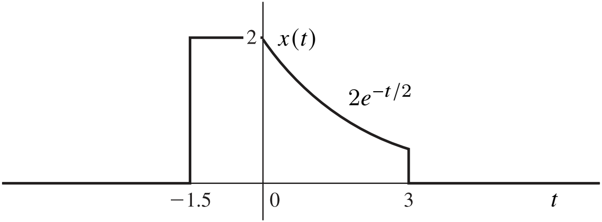
$$\mathrm{rect}(t) = \Pi(t) = u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2})$$
$$= \begin{cases} 1 & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & |t| \geq \frac{1}{2} \end{cases}$$



Piecewise functions and unit step

unit step can be used to describe piecewise functions using a single expression

Example

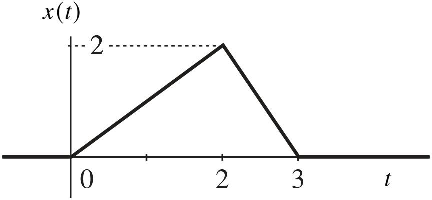


we can describe the signal $x(t)$ by a single expression valid for all t:

$$x(t) = \underbrace{2[u(t+1.5) - u(t)]}_{\text{constant part}} + \underbrace{2e^{-t/2}[u(t) - u(t-3)]}_{\text{exponential part}}$$

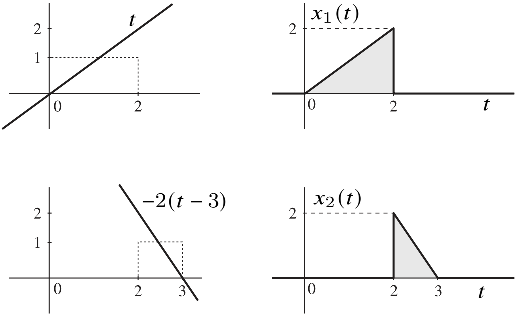
$$= 2u(t+1.5) - 2(1 - e^{-t/2})u(t) - 2e^{-t/2}u(t-3)$$

Example 1.8



describe the signal $x(t)$ using the unit step function

Solution:



using line equation $x = mt + b$ and unit step functions, the signal can represented as an addition of two components:

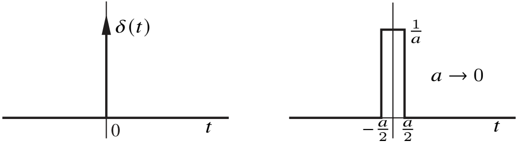
$$x_1(t) = t[u(t) - u(t-2)], \qquad x_2(t) = -2(t-3)[u(t-2) - u(t-3)]$$

therefore,

$$x(t) = x_1(t) + x_2(t) = tu(t) - 3(t-2)u(t-2) + 2(t-3)u(t-3)$$

Unit impulse

a *unit impulse* or (Dirac's) *delta function* $\delta(t)$ is an idealization of a signal that has unit area, very large near $t = 0$, and very small otherwise



- other forms of approximation can be used such as triangular; the shape is not important but the area is important
- $\delta(t)$ satisfies the property:

$$\delta(t) = 0, \quad t \neq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

  undefined at $t = 0$ (not mathematically rigorous)

Properties of the impulse function

Product with impulse: for any function $g(t)$ continuous at t_0,

$$g(t)\delta(t - t_0) = g(t_0)\delta(t - t_0)$$

Sampling (sifting) property

$$\int_{t_1}^{t_2} g(t)\delta(t - t_0)dt = g(t_0) \qquad t_1 < t_0 < t_2$$

(here, the impulse is defined as a *generalized function* (distribution), which is a function defined by its effect on other functions)

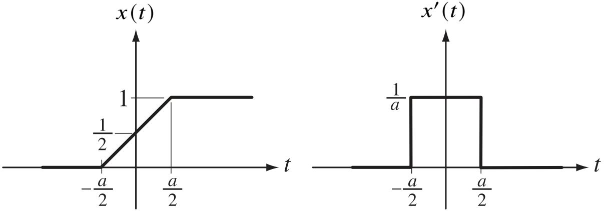
Scaling property

$$\delta\big(a(t - t_0)\big) = \frac{1}{|a|}\delta(t - t_0)$$

Unit impulse and step relation

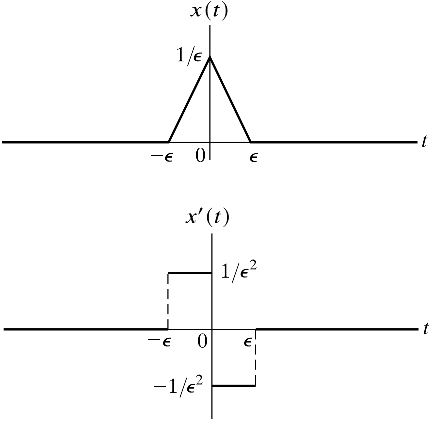
$$\frac{d}{dt}u(t-t_0) = \delta(t-t_0) \quad \text{and} \quad u(t-t_0) = \int_{-\infty}^{t} \delta(\tau - t_0)d\tau$$

Intuition



- as $a \to 0$ $x(t) \to u(t)$ and $x'(t) \to \delta(t)$
- $\delta(t)$ is called the *generalized derivative* of $u(t)$

The first derivative of the impulse function



- $x(t)$ is an impulse-generating function: $x(t) \rightarrow \delta(t)$ as $\epsilon \rightarrow 0$
- $x'(t)$ is derivative of this impulse-generating function
- $\delta'(t)$ is defined as $x'(t)$ as $\epsilon \rightarrow 0$; ($\delta'(t)$ is called a *moment* or *unit doublet* function)

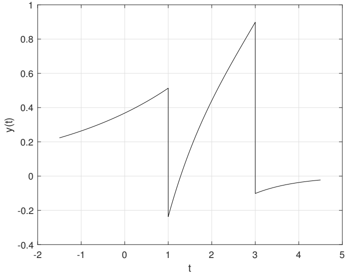
Matlab plotting example

for $x(t) = e^{-t}u(t)$, the following Matlab code plots

$$y(t) = x(\tfrac{-t+3}{3}) - (3/4)x(t-1) \quad \text{over} \quad -1.5 \le t \le 4.5$$

Matlab code

```
u = @(t) 1.0*(t>0);
x = @(t) exp(-t).*u(t); y = @(t) x((-t+3)/3)-3/4*x(t-1);
t = (-1.5:.0001:4.5); plot(t,y(t),'k');
xlabel('t'); ylabel('y(t)'); grid on;
```



Outline

- continuous-time signals
- signal operations
- useful CT signals
- **even and odd signals**
- signal energy and power

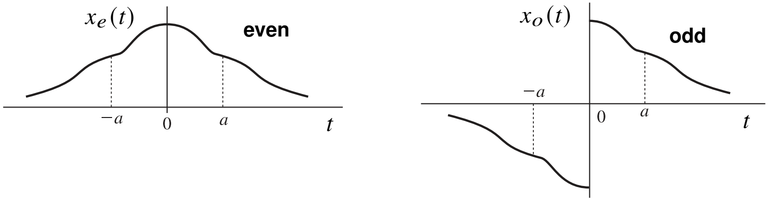
Even and odd signals

Even signal: an *even signal* $x_e(t)$ is symmetrical about the vertical axis

$$x_e(t) = x_e(-t)$$

Odd signal: an *odd signal* $x_o(t)$ is antisymmetrical about the vertical axis

$$x_o(t) = -x_o(-t)$$



Properties

$$\text{even function} \times \text{even function} = \text{even function}$$
$$\text{odd function} \times \text{odd function} = \text{even function}$$
$$\text{even function} \times \text{odd function} = \text{odd function}$$

Area

- for even functions
$$\int_{-a}^{a} x_e(t)dt = 2\int_{0}^{a} x_e(t)dt$$
- for odd function
$$\int_{-a}^{a} x_o(t)dt = 0$$

(under the assumption that there is no impulse at the origin)

Even and odd decomposition

every signal $x(t)$ can decomposed into an even and odd components:

$$x(t) = \underbrace{\tfrac{1}{2}[x(t) + x(-t)]}_{\text{even part}} + \underbrace{\tfrac{1}{2}[x(t) - x(-t)]}_{\text{odd part}}$$

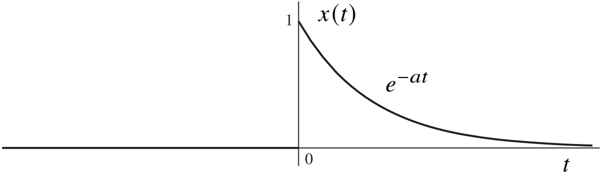
Examples

- $e^{jt} = x_e(t) + x_o(t)$ with

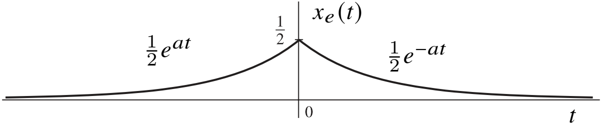
$$x_e(t) = \tfrac{1}{2}[e^{jt} + e^{-jt}] = \cos t \qquad x_o(t) = \tfrac{1}{2}[e^{jt} - e^{-jt}] = j \sin t$$

- $x(t) = e^{-at}u(t) = x_e(t) + x_o(t)$ with

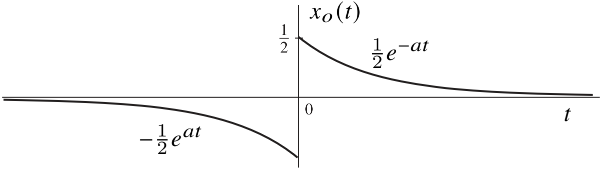
$$x_e(t) = \tfrac{1}{2}[e^{-at}u(t) + e^{at}u(-t)]$$
$$x_o(t) = \tfrac{1}{2}[e^{-at}u(t) - e^{at}u(-t)]$$


$x(t)$
1
e^{-at}
0
t

(a)


$x_e(t)$
$\frac{1}{2}$
$\frac{1}{2}e^{at}$
$\frac{1}{2}e^{-at}$
0
t

(b)


$x_o(t)$
$\frac{1}{2}$
$\frac{1}{2}e^{-at}$
0
t
$-\frac{1}{2}e^{at}$

(c)

Complex signal decomposition

a signal is **complex** if it has the form $x(t) = x_{\rm r}(t) + jx_{\rm im}(t)$

Conjugate-symmetric: $x(t)$ is *conjugate-symmetric* or *Hermitian* if

$$x(t) = x^*(-t)$$

Conjugate-antisymmetric: $x(t)$ is *conjugate-antisymmetric* or *skew Hermitian* if

$$x(t) = -x^*(-t)$$

- conjugate-symmetric signals have even real part and odd imaginary part
- conjugate-antisymmetric signals have odd real part and even imaginary part

any signal $x(t)$ can be decomposed into

$$x(t) = x_{\rm cs}(t) + x_{\rm ca}(t)$$

- $x_{\rm cs}(t) = \frac{1}{2}(x(t) + x^*(-t))$ is the conjugate-symmetric part
- $x_{\rm ca}(t) = \frac{1}{2}(x(t) - x^*(-t))$ is the conjugate-antisymmetric part

Exercise

determine the conjugate-symmetric and conjugate-antisymmetric components of

(a) $x_a(t) = e^{jt}$

(b) $x_b(t) = je^{jt}$

(c) $x_c(t) = \sqrt{2}e^{j(t+\pi/4)}$

Outline

- continuous-time signals
- signal operations
- useful CT signals
- even and odd signals
- **signal energy and power**

Signal energy and power

Energy of a signal

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- finite if $|x(t)| \to 0$ as $|t| \to \infty$
- infinite otherwise

(average) Power of a signal

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

- P_x is the time average (mean) of $|x(t)|^2$
- $\sqrt{P_x}$ is the *rms* (root-mean-square) value of $x(t)$

Energy and power signals

an **energy signal** is a signal with finite energy

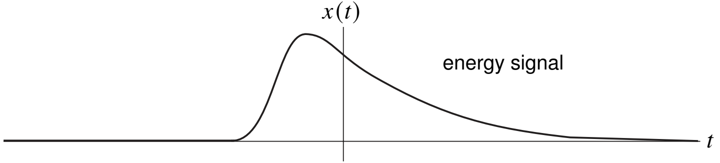
a **power signal** is a signal with finite and nonzero power

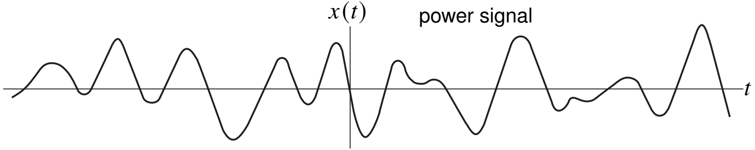
- an energy signal has zero power
- a power signal has infinite energy
- some signals are neither energy nor power signals

Power of periodic signals: a periodic signal $x(t)$ with period T_0 has power

$$P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{T_0}\int_{a_0}^{a_0+T_0} |x(t)|^2 dt$$

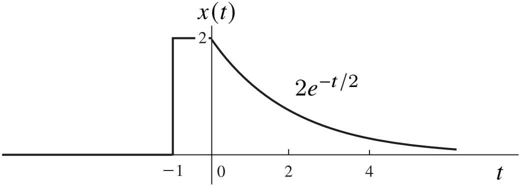
(not all power signals are periodic)


$x(t)$
energy signal
t

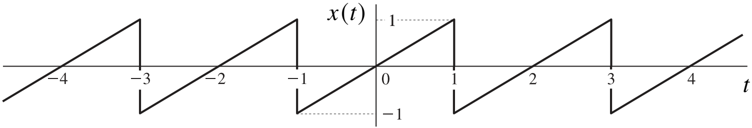

$x(t)$
power signal
t

Example 1.9

determine if the given signals are energy or power signals and find their energy/power



(a)



(b)

Solution:

(a) $|x(t)|$ goes to zero as $|t| \to \infty$, hence it is an energy signal with energy

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-1}^{0} 4dt + \int_{0}^{\infty} 4e^{-t} dt = 4 + 4 = 8$$

and $P_x = 0$

(b) $|x(t)|$ does not go to zero as $|t| \to \infty$, but it is periodic with period $T_0 = 2$, hence it is a power signal with power

$$\begin{aligned} P_x &= \frac{1}{T_0} \int_{a_0}^{a_0+T_0} |x(t)|^2 dt \\ &= \frac{1}{2} \int_{-1}^{1} |x(t)|^2 dt = \frac{1}{2} \int_{-1}^{1} t^2 dt = \frac{1}{3} \end{aligned}$$

the rms value of this signal is $1/\sqrt{3}$ and $E_x = \infty$

Example 1.10

determine the power and rms value of

(a) $x(t) = A\cos(\omega_0 t + \theta)$

(b) $x(t) = De^{j\omega_0 t}$

Solution

(a) the power is

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \cos^2(\omega_0 t + \theta) dt$$
$$= \lim_{T\to\infty} \frac{A^2}{2T} \int_{-T/2}^{T/2} [1 + \cos(2\omega_0 t + 2\theta)] dt = \frac{A^2}{2} + 0 = \frac{A^2}{2}$$

the zero term is because integral over a sinusoid is at most the area over half the cycle; thus dividing by T and letting $T \longrightarrow \infty$ gives zero

(a) alternative solution: we can also integrate over the period $T_0 = 2\pi/\omega_0$:

$$
\begin{aligned}
P_x &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A^2\cos^2(\omega_0 t + \theta)dt \\
&= \frac{A^2}{2T_0}\int_{-T_0/2}^{T_0/2}[1+\cos(2\omega_0 t + 2\theta)]dt = \frac{A^2}{2} + 0 = \frac{A^2}{2}
\end{aligned}
$$

– second term is zero because the integration of a sinusoid over a period is zero

– the rms value is $A/\sqrt{2}$

(b)

$$
P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|De^{j\omega_0 t}|^2 dt = \lim_{T\to\infty}\frac{|D|^2}{T}\int_{-T/2}^{T/2} dt = |D|^2
$$

Power of sum of two sinusoids

$$x(t) = A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2)$$

- if $\omega_1 \neq \omega_2$, then the power is

$$P_x = (A_1^2 + A_2^2)/2$$

- if $\omega_1 = \omega_2$, then the power is

$$P_x = \big(A_1^2 + A_2^2 + 2A_1 A_2 \cos(\theta_1 - \theta_2)\big)/2$$

Proof:

$$\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} [A_1\cos(\omega_1 t+\theta_1) + A_2\cos(\omega_2 t+\theta_2)]^2 dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A_1^2\cos^2(\omega_1 t+\theta_1) dt + \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A_2^2\cos^2(\omega_2 t+\theta_2) dt \\
&\quad + \lim_{T\to\infty} \frac{2A_1A_2}{T}\int_{-T/2}^{T/2} \cos(\omega_1 t+\theta_1)\cos(\omega_2 t+\theta_2) dt \\
&= \frac{A_1^2}{2} + \frac{A_2^2}{2} + \lim_{T\to\infty} \frac{2A_1A_2}{T}\int_{-T/2}^{T/2} \cos(\omega_1 t+\theta_1)\cos(\omega_2 t+\theta_2) dt
\end{aligned}$$

using

$$\begin{aligned}
&2\cos(\omega_1 t+\theta_1)\cos(\omega_2 t+\theta_2) \\
&\quad = \cos((\omega_1+\omega_2)t + \theta_1+\theta_2) + \cos((\omega_1-\omega_2)t + \theta_1-\theta_2)
\end{aligned}$$

- third term is zero if $\omega_1 \neq \omega_2$
- third term is $A_1A_2\cos(\theta_1-\theta_2)$ if $\omega_1 = \omega_2$

Power of sum of sinusoids

- the power of

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\omega_n t + \theta_n)$$

with *distinct* frequencies and $\omega_n \neq 0$ is

$$P_x = A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2$$

- the power of

$$x(t) = \sum_{k=m}^{n} D_k e^{j\omega_k t}$$

with *distinct* frequencies is

$$P_x = \sum_{k=m}^{n} |D_k|^2$$

Proof:

$$
\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t)dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=m}^{n}\sum_{\ell=m}^{n} D_k D_\ell^* e^{(j\omega_k-\omega_\ell)t}dt
\end{aligned}
$$

- the integrals of the cross-product terms (when $k \neq \ell$) are finite because the integrands are periodic signals (made up of sinusoids)
- these terms, when divided by $T \to \infty$, yield zero
- the remaining terms $(k = \ell)$ yield

$$
P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=m}^{n} |D_k|^2\, dt = \sum_{k=m}^{n} |D_k|^2
$$

Remarks

- in signal processing, when approximating $x(t)$ by $\hat{x}(t)$, the *error* is defined as

$$e(t) = x(t) - \hat{x}(t)$$

  the energy (or power) of $e(t)$ serves as a measure of the approximation's quality

- in communication systems, signals can be corrupted by noise during transmission; the quality of received signal is assessed by signal-to-noise power ratio

- the units of energy and power vary based on the signal type:
  - for a voltage signal $x(t)$, the energy E_x has units of volts squared-seconds ($\mathrm{V}^2\mathrm{s}$), and the power P_x has units of volts squared
  - for a current signal $x(t)$, the units are amperes squared-seconds ($\mathrm{A}^2\mathrm{s}$) for energy and amperes squared for power

Matlab example

use Matlab to approximate the energy of $x(t) = e^{-t}\cos(2\pi t)u(t)$

```
x = @(t) e^(-t).*cos(2 *pi *t).*u(t);
x_squared = @(t) x(t).*x(t);
t = 0:0.001:100;
dx=0.001;
Ex = sum(x_squared(t)*dx)
```

[output: `Ex = 0.2567`]

a better approximation can be obtained with the `quad` function

```
Ex = quad(x_squared,0,100)
```

[output: `Ex = 0.2562`]

Exercise: use Matlab to confirm that th energy of

$$y(t) = x(2t+1) + x(-t+1)$$

is $E_y = 0.3768$

References

- B.P. Lathi, *Linear Systems and Signals*, Oxford University Press, chapter 1 (1.1–1.5)
- M. J. Roberts, *Signals and Systems: Analysis Using Transform Methods and MATLAB*, McGraw Hill, chapter 2